Hermite-hadamard Type Inequalities for Harmonic-arithmetically Extended s-ε-Convex Functions

Ying Zheng*
Academic Periodicals Agency, Inner Mongolia University for Nationalities, Tongliao 028043, China
*Corresponding author

Abstract—In the paper, the authors introduce a new concept of harmonic-arithmetically extended s-ε-convex functions and establish some inequalities of Hermite-Hadamard type for this class of functions.

Keywords—harmonic-arithmetically extended s-ε-convex function; Hermite-Hadamard type integral inequalities

I. INTRODUCTION

Definition 1.1
A function \( f : I \subseteq R \rightarrow R \) is said to be convex if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \).

Definition 1.2 Let \( X \) be a real linear space, \( D \subseteq X \) is convex set, and \( \varepsilon \geq 0 \). A function \( f : D \rightarrow R \) is said to be \( \varepsilon \)-convex function on \( D \) if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon,
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \).

The concept of \( s \)-convex function was introduced in article [2]:

Definition 1.3 Let \( s \in (0,1] \) be a real number. A function \( f : I \subseteq R \rightarrow R \) is said to be \( s \)-convex if
\[
f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \).

The concept of generalized \( s \)-convex function was introduced in article [3]:

Definition 1.4 For some \( s \in [-1,1] \)

A function \( f : I \subseteq R \rightarrow R \) is said to be extended \( s \)-convex function if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
holds for all \( x, y \in I \) and \( t \in (0,1) \).

Definition 1.5 Let \( m \in (0,1) \). A function \( f : (0,b] \rightarrow R \) is said to be harmonic-arithmetically \( m \)-convex function if
\[
f\left(\frac{t+\frac{1-t}{m}}{x+y}\right) \leq tf(x) + mf(1-t)f(y)
\]
holds for all \( x, y \in (0,b] \) and \( t \in [0,1] \).

Definition 1.6 For some \( s \in [-1,1] \), a function \( f : I \subseteq R \rightarrow R \) is said to be harmonic-arithmetically extended \( s \)-convex if
\[
f\left(\frac{t+\frac{1-t}{s}}{x+y}\right) \leq tf(x) + (1-t)f(y)
\]
holds for all \( x, y \in I \) and \( t \in (0,1) \). If the inequality (6) is reversed, then \( f \) is said to be harmonic-arithmetically extended \( s \)-concave function. Study of convex functions and the Hermite-Hadamard type integral inequalities have always been a very active research topic. First, in article [5], S. S. Dragomir give the Hermite-Hadamard type integral inequalities of convex functions, as follows:

Theorem 1.1[6, Theorem 2.2 and 2.3]: Let \( f : I \subseteq R \rightarrow R \) be a
differentiable on \( I \) and \( a, b \in I^{o} \), with \( a < b \).

(i) If \( |f'| \) is convex on \([a, b]\), then

\[
\left| f(a) + f(b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{(b-a)|f'(a)| + |f'(b)|}{8};
\]

(ii) If \( |f'|^{p} \) is convex on \([a, b]\) for \( p > 1 \), then

\[
\left| f(a) + f(b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{2(p+1)} \left( |f'(a)|^{p} + |f'(b)|^{p} \right)^{1/p}.
\]

In article [7], U.S. Kirmaci proved the following inequality is established:

**Theorem 1.2** ([7. Theorem 2.3 and 2.4]) Let \( f : I \subseteq R \to R \) be a differentiable on \( I \) and \( a, b \in I^{o} \) with \( a < b \). If \( |f'|^{p} \) is convex on \([a, b]\) for \( p > 1 \), then

\[
\left| f(a) + f(b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{2(p+1)} \left( |f'(a)|^{p} + |f'(b)|^{p} \right)^{1/p}.
\]

In article [8] gives the following Hermite-Hadamard type integral inequality of \( S^{\varepsilon} \) convex function:

**Theorem 1.3** ([7]) Let \( f : I \subseteq R \to R \) be differentiable on \( I^{o} \) and \( a, b \in I \) with \( a < b \). If \( |f'|^{p} \) is convex on \([a, b]\) for some fixed \( s \in (0,1) \) and \( q \geq 1 \), then

\[
\left| f(a) + f(b) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \left( \frac{b-a}{2} \right)^{s} \left( \frac{2+2^{s}}{(s+1)(s+2)} \right)^{1/p} \left( |f'(a)|^{p} + |f'(b)|^{p} \right)^{1/p}.
\]

The main purpose of this paper is to introduce the concept of “harmonic-arithmetically extended \( S^{\varepsilon} \) convex functions” to establish some new Hermite-Hadamard type inequalities for these classes of functions.

**II. DEFINITION AND LEMMA**

Now we introduce the concept of harmonic-arithmetically extended \( S^{\varepsilon} \) convex functions:

**Definition 2.1**

For some \( s \in [-1,1] \) and \( \varepsilon \geq 0 \), a function \( f : I \subseteq R \to R \) is said to be harmonic-arithmetically extended \( S^{\varepsilon} \) convex if

\[
f\left( \frac{t + (1-t)}{x} \right) \leq t f(x) + (1-t) f(y),
\]

holds for all \( x, y \in I \) and \( t \in (0,1) \). If the inequality (12) is reversed, then \( f \) is said to be harmonic-arithmetically extended \( S^{\varepsilon} \) concave function.

**Lemma 2.1** Let \( f : I \subseteq R \to R \) be a differentiable function on \( I \) with \( a, b \in I \), \( a < b \). If \( f' \in L_{1}([a,b]) \), then

\[
f(H(a,b)) - \frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^{s}} \, dx - \frac{b-a}{4a+b} \left( 1-t(\frac{a}{b})^{1-t} + (1-t)(|H(a,b)|)^{1-t} \right) \frac{(b-a)}{(b-a)} \int_{a}^{b} \frac{f(x)}{x^{s}} \, dx.
\]

where.

\[
H(a,b) = \frac{2ab}{a+b}
\]

**Proof.** Let \( x = \left( ta^{s} + (1-t)[H(a,b)]^{-1} \right) \) for \( t \in [0,1] \), then

\[
\int_{a}^{b} \left( ta^{s} + (1-t)[H(a,b)]^{-1} \right)^{-1} f'( \left( ta^{s} + (1-t)[H(a,b)]^{-1} \right)) \, dx = \frac{2ab}{b-a} - \frac{2ab}{b-a} \int_{a}^{b} f(x) \, dx.
\]

Similarly, letting \( x = \left( tb^{-1} + (1-t)[H(a,b)] \right)^{-1} \) for all \( t \in [0,1] \), gives

\[
\int_{a}^{b} \left( tb^{-1} + (1-t)[H(a,b)]^{-1} \right)^{-1} f'( \left( tb^{-1} + (1-t)[H(a,b)]^{-1} \right)) \, dx = \frac{2ab}{b-a} \int_{a}^{b} f(x) \, dx.
\]

Adding these two equalities leads to Lemma 2.1.
Lemma 2.1 Let \( s > -1, u, v > 0 \), with \( u \neq v \), then
\[
H(u,v)\geq \frac{1}{b-a} \int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]
\[
\geq \frac{u+v}{b-a} \int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]
\[
\geq \frac{u+v}{b-a} \int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]

III. MAIN RESULTS

Theorem 3.1 Let \( s \in (-1,1) \) and \( \varepsilon \geq 0 \),
\[
f : I \subseteq R, \rightarrow R \text{ be differentiable on } I, a, b \in I \text{ with } a < b, f' \in L_{1}([a,b]),
\]
If \( f' \) is harmonic-arithmetically extended \( S^{c} \epsilon \)-convex on \([a,b] \)
\[
\text{for } q \geq 1, \text{ then }
\[
\int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]
\[
\geq \frac{u+v}{b-a} \int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]
\[
\geq \frac{u+v}{b-a} \int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]

where \( H(u,v), S(u,v,s), T(u,v,s) \) is defined by lemma 2.2.

Proof. Since Lemma 2.1 and Hölder inequality, then
\[
\left| f(H(a,b)) - \frac{ab}{b-a} \int_{a}^{b} f(x) dx \right|
\]
\[
\leq \frac{b-a}{4a} \left[ \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt \right]^{rac{1}{q}}
\]
\[
\leq \left[ \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt \right]^{rac{1}{q}}
\]
\[
\leq \left[ \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt \right]^{rac{1}{q}}
\]

Using Lemma 2.2, then
\[
\int_{a}^{b} \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt
\]

By the harmonic-arithmetically extended \( S^{c} \epsilon \)-convexity of function \( f' \) on \([a,b] \), fundamental inequality and Lemma 2.2, then following (16) and (17)

A combination of (15) to (17) gives the required inequality (14).

Corollary 3.1.1 Under the assumptions of Theorem 3.1, when \( q = 1 \), we have
\[
\left| f(H(a,b)) - \frac{ab}{b-a} \int_{a}^{b} f(x) dx \right| \leq \frac{b-a}{240ab} \left| S(a, H(a,b), s) f'(a) \right|
\]

Corollary 3.1.2 Under the assumptions of Theorem 3.1, if \( q = 1 \) and \( s = 1 \), then
\[
\left| f(H(a,b)) - \frac{ab}{b-a} \int_{a}^{b} f(x) dx \right| \leq \frac{b-a}{240ab} \left[ \left(1-t\right)^{s+1-\left(t-H(a,b)\right)^{2}} dt \right]
\]

Theorem 3.1.2 Let \( s \in (-1,1) \) and \( \varepsilon \geq 0 \),
\[
f : I \subseteq R, \rightarrow R \text{ be differentiable on } I, a, b \in I \text{ with }
\]

If $f''$ is harmonic-arithmetically extended $s$-$E$-convex on $[a, b]$ for $q > 1$, then

$$
\int_a^b f'(x)^q \, dx \leq (q-1)^{-\frac{q}{q-1}} \left[ \frac{1}{(s+1)(s+2)} \left( \int_a^b f'(x)^q \, dx \right)^{\frac{q}{q-1}} \right]
$$

(18)

Proof. Since Lemma 2.1 and Hölder inequality, then

$$
\left( \int_a^b f'(x)^q \, dx \right)^{\frac{q}{q-1}} \leq \left( \int_a^b f'(x)^{q-1} \, dx \right)^{\frac{q}{q-1}} \left( \int_a^b f'(x)^{\frac{q}{q-1}} \, dx \right)^{\frac{q}{q-1} - 1}
$$

(19)

A combination of (19) to (23) gives the required inequality (18).

REFERENCES


