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Option Pricing under the CEV Model in a Composite-diffusive Regime

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***Abstract*—Constant elasticity of variance model for option pricing in a composite-diffusive regime is established. We obtain the Black-Scholes differential equation and the corresponding Black-Scholes formula for the prices of European call option. Furthermore, we discuss an asymptotic expansion of the European call option price as the elasticity factor tends to 2.**

Keywords—pricing; CEV model; stable subordinator; asymptotic expansion

I. INTRODUCTION

The constant elasticity of variance (CEV, in short) model was introduced by[1, 2]. This model assumes that the dynamics of underlying stock price is governed by

$$dS_t = \mu S_t dt + \sigma S_t^{\beta/2} dB_t, \tag{1}$$

where B_t is a standard Brownian motion, μ is the rate of mean return of the stock, σ is the volatility, β is the elasticity factor. When $\beta = 2$ the CEV model degenerates to the classical Black-Scholes (BS, in short) model [3]. Empirical evidence (see [4-6]) has shown that the CEV diffusion process could be a better candidate for describing the actual stock price behaviour than the BS model. Moreover, The CEV model can capture implied volatility's smile or skew phenomena while the classical BS model cannot.

In [7], Magdziarz applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion (SGBM, in short)

$$X_\alpha(t) = X(S_\alpha(t)), \tag{2}$$

as the model of asset prices exhibiting subdiffusive dynamics. Here, $X_\alpha(t)$ is a subordinated process, in which $X(\tau)$ is the Geometric Brownian Motion (GBM, in short) and $S_\alpha(t)$ is the inversed α stable subordinator defined in the following way

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, 0 < \alpha < 1, \tag{3}$$

where $U_\alpha(\tau)$ is a strictly increasing α-stable Levy process (see,[8,9]).Based on the work of Magdziarz, scholars([11-14]) generalize the subdiffusuve Brownian motion model.Specially, Liang et al.[10] generalize Magdziarz's model to a composite-diffusive regime. They introduced a composite-diffusive geometric Brownian motion

$$X_{\alpha,H}(t) = X_H(S_\alpha(t)), \tag{4}$$

as the model of asset prices, where $X_H(\tau)$ is the Fractional Geometric Brownian Motion (FBM, in short).

In this paper, based on the work of Magdziarz [7] and Liang et al.[10], we will study the option pricing problem under the CEV model in a composite-diffusive regime, i.e., the underlying stock price $Z_t = X_H(S_\alpha(t))$ satisfies the following stochastic differential equation

$$dZ_t = \mu Z_t dS_\alpha(t) + \sigma Z_t^{\beta/2} db_H(S_\alpha(t)), \tag{5}$$

where $\mu,\ \sigma,\ \beta,\ Z_0$ are constants, and

$$db_H(\tau) = \omega(\tau)(d\tau)^H$$

is a modified fractional Brownian motion with Hurst exponent $H \in \left[\frac{1}{2}, 1\right]$, $\omega(\tau)$ is the companion Gaussian white noise with zero mean and unit variance. In particular, it is assumed that the $S_\alpha(t)$ is independent of $b_H(t)$.

In addition, Beckers[6] finds thirty-seven out of forty-seven stocks in a year daily data set to have estimated β to be less than 2. So, in this paper, we assume that $0 \le \beta < 0$.

The rest of the paper is organized as follows. In section 2, we obtain a Black-Scholes equation driven by $X_H(S_\alpha(t))$ defined by equation (2) and give the corresponding Black-Scholes formula for the option price of an European call option

for $\alpha H > \frac{1}{2}$. In Section 3, by using the perturbation theory (see, [15]) for PDE, we obtain an asymptotic representation of the European call option price as the elasticity factor β tends to 2.

II. Black-Scholes Equation and Formula

In this section, we follow the other usual assumptions of the classical BS model but with the exception that the underlying stock price satisfies (2).

Let $C(t, S_t)$ be the value at time t of a European call option with expiration $T < +\infty$ and exercise price K. We have the following theorem.

Theorem 1 Assume that the price of option $C(t, Z_t)$ belongs to $C^{1,2}([0,T)\times[0,+\infty))$, then $C(t, Z_t)$ satisfies the following partial differential equation

$$\frac{\partial C}{\partial t} + \frac{H\sigma^2}{\Gamma(\alpha)^{2H}} t^{2\alpha H-1} Z^{\beta} \frac{\partial^2 C}{\partial Z^2} + rZ\frac{\partial C}{\partial Z} - rC = 0, \quad (6)$$

with boundary condition

$$C(T, Z_T) = (Z_T - K)^+, \quad (7)$$

where r is the risk-free interest rate.

Proof : By the Taylor's formula, we have

$$C(t+\Delta t, Z+\Delta Z) = C(t,Z) + \frac{\partial C(t,Z)}{\partial t}\Delta t + \frac{\partial C(t,Z)}{\partial Z}\Delta Z + \frac{1}{2}\frac{\partial^2 C(t,Z)}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 C(t,Z)}{\partial Z^2}(\Delta Z)^2 + \frac{\partial^2 C(t,Z)}{\partial t \partial Z}\Delta t \Delta Z + o((\Delta t)^2 + (\Delta Z)^2) \quad (8)$$

From [10] we know that

$$\Delta Z = \mu Z \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Delta t + \sigma Z \Delta b_H(S_\alpha(t)) + o(\Delta t), \quad (9)$$

$$(\Delta Z)^2 = \sigma^2 Z^{\beta}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H} + O(\Delta t)^{\min(1,2H)}, \quad (10)$$

$$(\Delta Z)(\Delta t) = o(\Delta t). \quad (11)$$

We construct a portfolio Π of $C(t,Z)$ and Z such that

$$\Pi_t = C(t, Z(t)) - \Delta Z(t), \quad (12)$$

here, $\Delta = \frac{\partial C(t,Z)}{\partial Z}$ denotes the units of the stock, Π is the price of a riskless portfolio. Then, it follows from (8)-(11) that

$$d\Pi = \left[\frac{\partial C(t,Z)}{\partial t} + \sigma^2 H t^{2H-1}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H} Z^{\beta}\frac{\partial^2 C(t,Z)}{\partial Z^2}\right]dt.$$

If the portfolio is riskless during time t, then

$$d\Pi = r\Pi dt = r\left(C - \frac{\partial C(t,Z)}{\partial Z}Z\right)dt.$$

Thus, we can obtain that

$$\frac{\partial C}{\partial t} + \frac{H\sigma^2}{\Gamma(\alpha)^{2H}} t^{2\alpha H-1} Z^{\beta} \frac{\partial^2 C}{\partial Z^2} + rZ\frac{\partial C}{\partial Z} - rC = 0,$$

The proof is completed.

In the end of the section, we discuss the corresponding Black-Scholesformula. Denote

$$H' = \alpha H,\ \hat{\sigma} = \frac{\sigma}{\sqrt{\alpha}\Gamma(\alpha)^H}$$

then (6) changes into

$$\frac{\partial C}{\partial t} + \frac{H\sigma^2}{\Gamma(\alpha)^{2H}} t^{2\alpha H-1} Z^{\beta} \frac{\partial^2 C}{\partial Z^2} + rZ\frac{\partial C}{\partial Z} - rC = 0, \quad (13)$$

Theorem 2 Suppose that $H' > \frac{1}{2}$, then the solution of problem (13) and (7) ,i.e. the value of a European call option is given by

$$C(t,Z) = Z\sum_{n=0}^{\infty}\frac{\upsilon^n \exp(-\upsilon)}{\Gamma(n+1)} G(n+1+1/(2-\beta), \omega)$$

$$-K\exp(-r(T-t))\sum_{n=0}^{\infty}\frac{\upsilon^{n+1/(2-\beta)}\exp(-\upsilon)}{\Gamma(n+1+1/(2-\beta))}G(n+1,\omega)$$

where

$$\upsilon=\frac{Z^{2-\beta}\exp(r(2-\beta)(T-t))}{2c(t)},\omega=\frac{K^{2-\beta}}{2c(t)}$$

$$c(t)=\frac{1}{2}H'\hat{\sigma}^2\int_0^{\tau}(T-s)^{2H'-1}\exp(r(2-\beta)(T-s))ds,$$

$$\tau=T-t,G(\xi,\omega)=\frac{1}{\Gamma(\xi)}\int_{\omega}^{\infty}s^{\xi-1}\exp(-s)ds.$$

here $\Gamma(\xi)$ is the Gamma function.

From theorem 2, we can see that $c(t)$ cannot be determined analytically, therefore, the above pricing formula can not to be use directly. Hence we turn into find the asymptotic representation of the solution of the PDE problem (13) and (7).

III. Asymptotic Representation of Price $C(t,Z(t))$

In this section, we introduce a small positive parameter ε such that $\beta=2-\varepsilon$, where $0<\varepsilon<<1$. Then, we will obtain an asymptotic representation of the solution of the PDE problem (13) and (7) in the next theorem.

Theorem 3 Suppose the European call option price $C(t,Z)$ which is the solution of (13) and (7) has an asymptotic expansion price such as

$$C(t,Z)=C_0+\varepsilon C_1+\varepsilon^2C_2+\cdots. \tag{14}$$

Then $C_0(t,Z)$ with the final condition

$$C_0(t,Z)=(Z-K)^+$$

is given by

$$C_0(t,Z)=ZN(d_1)-K\exp(-r(T-t))N(d_2),$$

where

$$d_1=\frac{\ln(Z/K)+r(T-t)+\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H}-t^{2\alpha H})}{\sqrt{\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H}-t^{2\alpha H})}},$$

$$d_2=d_1-\sqrt{\frac{\sigma^2}{\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H}-t^{2\alpha H})},$$

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy,$$

and each $C_n(t,Z)$ with the final condition $C_n(t,Z)=0$ is recursively given by

$$C_n(t,Z)=e^{-r(T-t)}F(x,y),$$

$$F(x,y)=\frac{1}{2\sqrt{\pi}}\int_0^x\int_{-\infty}^{+\infty}\frac{\varphi(\xi,\eta)}{\sqrt{x-\xi}}e^{-\frac{(y-\eta)^2}{4(x-\xi)}}d\eta d\xi,$$

$$\varphi(x,y):=e^{-r(T-t)}\sum_{k=0}^{n-1}\frac{(-\hat{Z})^{n-k}}{(n-k)!}\left(\frac{\partial^2C_k}{\partial\hat{Z}^2}-\frac{\partial C_k}{\partial\hat{Z}}\right),$$

$$y=\hat{Z}+r(T-t)-\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2H}-t^{2H}),$$

$$x=-\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2H}-t^{2H}),$$

$$\hat{Z}=\ln Z.$$

Remark 1 In particular, letting $H=1/2$ and $\alpha\to1$, then Eq.(13) changes to

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2Z^{\beta}\frac{\partial^2C}{\partial Z^2}+rZ\frac{\partial C}{\partial Z}-rC=0,$$

Therefore, the proposed pricing model of the European call option can be treated as an extension of the model in [19].

Proof From $\hat{Z}=\ln Z$,the PDE problem (13) and (7) becomes

$$C_t+\frac{H\sigma^2}{\Gamma(\alpha)^{2H}}t^{2\alpha H-1}e^{-\varepsilon\hat{Z}}(C_{\hat{Z}\hat{Z}}-C_{\hat{Z}})+r(C_{\hat{Z}}-C)=0, \tag{15}$$

$$C(T,\hat{Z})=\left(e^{\hat{Z}}-K\right)^+. \tag{16}$$

Now, we define the partial differential operator L_0 by

$$L_0=\frac{\partial}{\partial t}+\frac{H\sigma^2}{\Gamma(\alpha)^{2H}}t^{2\alpha H-1}\left(\frac{\partial^2}{\partial\hat{Z}^2}-\frac{\partial}{\partial\hat{Z}}\right)+r\left(\frac{\partial}{\partial\hat{Z}}-\right).$$

Then, by using Taylor series of e^x, from (15-16) we can obtain

$$L_0 C_0 = 0, \tag{17}$$

$$L_0 C_1 = \frac{H\sigma^2}{\Gamma(\alpha)^{2H}} t^{2\alpha H-1} \hat{Z}\left(\frac{\partial^2 C_0}{\partial \hat{Z}^2} - \frac{\partial C_0}{\partial \hat{Z}}\right),$$

$$L_0 C_n = g_n(t, \hat{Z}), n \geq 1, \tag{18}$$

where

$$g_n(t, \hat{Z}) = -\frac{H\sigma^2}{\Gamma(\alpha)^{2H}} t^{2\alpha H-1} \sum_{k=0}^{n-1} \frac{(-\hat{Z})^{n-k}}{(n-k)!}\left(\frac{\partial^2 C_k}{\partial \hat{Z}^2} - \frac{\partial C_k}{\partial \hat{Z}}\right),$$

and the relevant final conditions are given by

$$C_0(T, \hat{Z}) = (e^{\hat{Z}} - K)^+, \tag{19}$$

$$C_n(T, \hat{Z}) = 0, n \geq 1. \tag{20}$$

After changing back to the original variables Z from $\hat{Z}$, the solution of (17) with the final condition (19) is given by

$$C_0(t, Z) = ZN(d_1) - K\exp(-r(T-t))N(d_2),$$

where

$$d_1 = \frac{\ln(Z/K) + r(T-t) + \frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H} - t^{2\alpha H})}{\sqrt{\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H} - t^{2\alpha H})}},$$

$$d_2 = d_1 - \sqrt{\frac{\sigma^2}{\alpha\Gamma(\alpha)^{2H}}(T^{2\alpha H} - t^{2\alpha H})},$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

In addition, to obtain the correction terms C_n, we apply the following transformation variables

$$y = \hat{Z} + r(T-t) - \frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2H} - t^{2H}), \tag{21}$$

$$x = -\frac{\sigma^2}{2\alpha\Gamma(\alpha)^{2H}}(T^{2H} - t^{2H}), \tag{22}$$

and

$$F(x, y) = C_n(t, \hat{Z}) e^{r(T-t)}, \tag{23}$$

to (18) and (20) yields

$$\frac{\partial F}{\partial x} - \frac{\partial^2 F}{\partial y^2} = \varphi(x, y), \tag{24}$$

$$F(0, y) = 0, \tag{25}$$

where

$$\varphi(x, y) := e^{-r(T-t)} \sum_{k=0}^{n-1} \frac{(-\hat{Z})^{n-k}}{(n-k)!}\left(\frac{\partial^2 C_k}{\partial \hat{Z}^2} - \frac{\partial C_k}{\partial \hat{Z}}\right).$$

According to heat equation theory,the solution to(24-25) and is given by

$$F(x, y) = \frac{1}{2\sqrt{\pi}} \int_0^x \int_{-\infty}^{+\infty} \frac{\varphi(\xi, \eta)}{\sqrt{x-\xi}} e^{-\frac{(y-\eta)^2}{4(x-\xi)}} d\eta d\xi,$$

since each g_n is determined by C_i $i = 0,1\cdots, n-1$,therefore is determined recursively.

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