

Multi-objective programming problems with equilibrium constraints

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Abstract. In this paper, we solve multi-objective programming problems with equilibrium constraints by means of homotopy interior point algorithm. We also prove the global convergence of this homotopy interior point algorithm under assumptions. Moreover, the results of the numeric example shows that this method is feasible and effective.

Introduction

Given functions $f: R^{n+m} \rightarrow R^p, g: R^{n+m} \rightarrow R^s, G: R^{n+m} \rightarrow R^l, F: R^{n+m} \rightarrow R^m$. In this paper, we are devoted to the study of the multiobjective optimization problems with equilibrium constraints(MOPECS):

$$\begin{aligned} & \min f(x, y), \\ & \text{(MOPECS)} \quad s.t. Z \subseteq R^{n+m}, \\ & \quad y \in S(x), \end{aligned} \quad (1)$$

where $Z = \{(x, y) \in R^{n+m} : g(x, y) \leq 0\}$ is a nonempty closed convex set.

$$y \in S(x) \Leftrightarrow F(x, y)^T(v - y) \geq 0, C(x) = \{y \in R^m : G(x, y) \leq 0\}, X = \{x \in R^n : (x, y) \in Z, y \in R^m\}.$$

For $x \in X$, $S(x)$ is the solution set of a parametric variational inequality problem.

$$\text{(PVI)} \quad y \in S(x) \Leftrightarrow F(x, y)^T(v - y) \geq 0, \forall v \in C(x). \quad (2)$$

Where, for some $y \in R^m$, $X = \{x \in R^n : (x, y) \in Z\}$.

$(H_1) \forall x \in X, i = 1, 2, \dots, l, G_i(x, \cdot)$ is a convex function in the second argument;

$(H_2) \forall (x, y) \in \Omega, \nabla_y F(x, y) + \sum_{i=1}^l (\nabla_{yy}^2 G_i(x, y) + \nabla_y G_i(x, y)^T)$ is positive define, where

$$\Omega = \{(x, y) \in R^{n+m} : g(x, y) \leq 0, G(x, y) \leq 0\};$$

(H_3) 对于 $t \in (0, 1], q \in \Omega_2(t), \{\nabla g_i(q), i \in I(q), \nabla_q h(q, t)\}$ is full column rank, where

$$I(q) = \{i \in \{1, 2, \dots, s\} : g_i(q) = 0\}.$$

The KKT system of (2) as follows:

$$\begin{cases} F(x, y) + \nabla_y G(x, y)u = 0, \\ u \geq 0, G(x, y) \leq 0, UG(x, y) = 0, \end{cases} \quad (3)$$

where $U = doag(u)$.

The problem(1) is equivalent to:

$$\begin{aligned} \min & f(x, y), \\ \text{s.t.} & g(x, y) \leq 0, \\ & F(x, y) + \nabla_y G(x, y)u = 0, \\ & u \geq 0, G(x, y) \leq 0, UG(x, y) = 0. \end{aligned} \quad (4)$$

We construct the following homotopy equation

$$h(q, t) = \begin{pmatrix} F(x, y) + \nabla_y G(x, y)u \\ UG(x, y) + te \end{pmatrix} = 0,$$

where $q = (x, y, u)^T, e = (1, 1, \mathbf{L}, 1)^T \in R^l, t \in (0, 1]$.

Let $f(q) = f(x, y, u), g(q) = g(x, y, u)$, the problem(1) is given by^[7]:

$$\begin{aligned} \min & f(q), \\ \text{s.t.} & g(q) \leq 0, \\ & h(q) = 0, \end{aligned} \quad (5)$$

when $t \rightarrow 0$, the problem(5) is equal to (1). In the following, we solve the (5).

Preparation

Assumption:

(H_4) Ω is nonempty, bounded, connected;

(H_5) $h_i(q)$ is positive linear independent about $\nabla g(q)$;

(H_6) Ω holds on weak quasi-normal cone condition, there is a nonempty set $\Omega_1^0 \subset \Omega$ and positive linear independent mapping $h_i(q), i \in \{1, 2, \mathbf{L}, m\}$

$$\left\{ x + \sum_{i \in I(x)} u_i h_i(q) \mid u_i \geq 0 \right\} \cap \Omega_1^0 = \Phi, \forall q \in \partial \Omega.$$

Main results

To solve the KKT system, we construct a homotopy equation as follows:

$$H(w, w^{(0)}, t) = \begin{pmatrix} (1-t)(\nabla f(x)I + \nabla g(x)u + \nabla_q h(q, t)v + th(q)u^2) + t(q - q^{(0)}) \\ h(q, t) \\ Ug(q) - tU^{(0)}g(q^{(0)}) \\ (1-t)(1 - \sum_{i=1}^p I_i)e - t(I^{\frac{9}{4}} - (I^{(0)})^{\frac{9}{4}}) \end{pmatrix} = 0. \quad (6)$$

When $t = 1$,

$$\nabla_q h(q, t)b + (q - q^{(0)}) = 0;$$

$$h(q, 1) = 0;$$

$$Ug(x) - U^{(0)}g(x^{(0)}) = 0;$$

$$I^{\frac{9}{4}} - (I^{(0)})^{\frac{9}{4}} = 0.$$

If $b \neq 0$, contradicts to quasi-normal cone condition. So $q = q^{(0)}$.

When $t = 0$, the homotopy equation turns to the KKT system. When $w^{(0)}$ is given, Let $H(w, w^{(0)}, t) = H_{w^{(0)}}(w, t) \cdot H_{w^{(0)}}^{-1}(0) = \{(w, t) \in \Omega(t) \times (0, 1] : H_{w^{(0)}}(w, t) = 0\}$.

Theorem 2.1 Suppose that $H_{w^{(0)}}(w, t)$ is defined as in (6), f, g and h are two times continuously differentiable functions, Assumptions $(H_1) - (H_6)$ hold. Then, for almost all initial points $(q^{(0)}, I^{(0)}, u^{(0)}, v^{(0)})^T \in \Omega_1^0(1) \times \Lambda^{++} \times R_{++}^m \times \{0\}$, 0 is a regular value of $H_{w^{(0)}}$, and consists of some smooth curves. In addition, there is a smooth curve noted by $\Gamma_{w^{(0)}}$, which is starting from $(w^{(0)}, 1)$.

Lemma 2.2 Suppose that $(H_3) - (H_4)$, for all initial points $(q^{(0)}, I^{(0)}, u^{(0)}, v^{(0)})^T \in \Omega_1^0(1) \times \Lambda^{++} \times R_{++}^m \times \{0\}$, If 0 is a regular value of $H_{w^{(0)}}$, then the projection of the smooth curve $\Gamma_{w^{(0)}}$ on the component I is bounded.

Theorem 2.3 Suppose that $H_{w^{(0)}}(w, t)$ be defined as (6), f, g and h be two times continuously differentiable functions, and Assumptions $(H_1) - (H_6)$ hold. For almost all the initial point $(q^{(0)}, I^{(0)}, u^{(0)}, v^{(0)})^T \in \Omega_1^0(1) \times \Lambda^{++} \times R_{++}^m \times \{0\}$, if 0 is a regular value of $H_{w^{(0)}}$, then the curve $\Gamma_{w^{(0)}} \in \Omega(t) \times (0, 1]$ is bounded.

Theorem 2.4 Suppose that f, g, F are two times continuously differentiable functions, and G is triply continuously differentiable, Assumption $(H_1) - (H_6)$ hold. Then when $t \rightarrow 0$, the solution of the KKT system, and for almost all the initial point $(q^{(0)}, I^{(0)}, u^{(0)}, v^{(0)})^T \in \Omega_1^0(1) \times \Lambda^{++} \times R_{++}^m \times \{0\}$, $H_{w^{(0)}}^{-1}(0)$ contains a curve $\Gamma_{w^{(0)}}(w^{(0)}, 1)$. When $t \rightarrow 0$, the limit set Γ of $\Gamma_{w^{(0)}}$ is nonempty, and every point of Γ is the solution of the KKT system.

Proof: By theorem 2.1, for almost all the initial point $(q^{(0)}, I^{(0)}, u^{(0)}, v^{(0)})^T \in \Omega_1^0(1) \times \Lambda^{++} \times R_{++}^m \times \{0\}$, 0 is a regular value of $H_{w^{(0)}}$, and $H_{w^{(0)}}^{-1}(0)$ consists of some smooth curves. Among them, there is a smooth curve noted by $\Gamma_{w^{(0)}}$ which starting from $(w^{(0)}, 1)$.

By the classification theorem of one dimensional smooth manifolds, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or unit interval.

$$\text{Because } \frac{\partial H_{w^{(0)}}(w,1)}{\partial w} \Big|_{w=w^{(0)}} = \begin{pmatrix} \sum_{i=1}^{m+l} b_i \nabla_q^2 h(q^{(0)},1) + I_{m+n+l} & 0 & 0 & \nabla h(q^{(0)},1) \\ \nabla h(q^{(0)},1)^T & 0 & 0 & 0 \\ A^{(0)} \nabla g(q^{(0)})^T & 0 & \text{diag}(g(q^{(0)})) & 0 \\ 0 & -\frac{9}{4}(I^{(0)})^{\frac{5}{4}} I_p & 0 & 0 \end{pmatrix},$$

$g(q^{(0)}) < 0$, it is easy to know that $\partial H_{w^{(0)}}(w,1)/\partial w$ is non-singular. So, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit interval.

Let the limit point of $\Gamma_{w^{(0)}}$ be (w^*, t^*) , the following three cases are possible:

- 1) $(w^*, t^*) \in \Omega(t^*) \times \{1\}$;
- 2) $(w^*, t^*) \in \Omega(t^*) \times \{0\}$;
- 3) $(w^*, t^*) \in \partial\Omega(t^*) \times (0,1)$.

In the set $\Omega(1) \times \{1\}$, $H_{w^{(0)}}(w,1) = 0$ has a unique possible case $(w^{(0)}, 1)$. By theorem 2.3, case 3) will not happen. So, case 2) is the unique possible case and w^* is the solution of the KKT system

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