

Derivations on Trellises

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Abstract---In this paper, we introduce the notion of derivations for a trellis and investigate some related properties of this subject. We give some equivalent conditions under which a derivation is isotone for trellises. Also, we study xed points and de ne f -derivation on T and cartesian derivation on $T_1 \times T_2$

Keywords---trellis, isotone, mathematics

I. INTRODUCTION

In 1971, H. Skala introduced the notions of pseudo-ordered sets and trellises. Trellises are generalization of lattices by considering sets with a reflexive and antisymmetric, but not necessarily transitive. They are also an extension of lattices by postulating the existence of least upper bounds and greatest lower bounds on pseudo-ordered sets similarly as for partially ordered sets. Any reflexive and antisymmetric binary relation E on a nonempty set T is called a *pseudo - ordered* on T and $(T; E)$ is called a *pseudo - order set* or *psoset*. Clearly, each partial order is a pseudo-order. A natural example of a pseudo-order on the set of real numbers is obtained by setting xEy if and only if $0 \leq y - xa$ for a xed positive number a : Two elements x, y are comparable if xEy or yEx . For a subset L of T , the notions of a lower bound, and upper bound, the greatest lower bound $(g:l:b)$, the least upper bound $(l:u:b)$ are de ned analogously to the corresponding notions in a posets. Generally the notion of a derivation introduced in algebraic systems such as rings, near-rings, specially in [5]. Some properties of a derivation such as isotone of a derivation, the set of xed points of a derivation and relation of derivations with meet-translation as studied by G. Szasz [5]. Derivations on trellises already de ned by Shashirekha B. Rai, S. Parameshwara Bhatta in [2] with extra conditions that is made derivations isotone. Many authors investigate other properties of derivations on trellises and other algebraic systems in [1,2,3,5]. Here we introduced the notion of a derivation on trellises with weak conditions. The remainder of

this paper is organized as follows. In the second section we review the definitions and important theorems of the trellis. In section 3, an equivalent condition is given for a trellis in term of the derivation. Also, we study xed points and de ne f -derivation on T and cartesian derivation on $T_1 \times T_2$:

II. PRELIMINARIES

Definition. Let T be a nonempty set. A trellis is a psoset $T; E$ where any two of whose elements have a $(g:l:b)$ and a $(l:u:b)$. Any psoset can be regarded as a diagram (possibly infinite) in which for any pair of distinct points u and v either there is no directed line between u and v , or if there is a directed line from u to v , there is no directed line from v to u . **Definition.** A trellis T is *associative* if the following conditions hold for all

$$x, y, z \in T; \quad (x \wedge y) \wedge z = x \wedge (y \wedge z) \text{ or } (x \vee y) \vee z = x \vee (y \vee z);$$

Example 2.1. The psoset $A = \{0; a; b; c; 1\}$ with $0 E a E b E c E 1, 0 E x E 1$ for every $x \in \{a; b; c\}$ and $0 E 1$ while a and c are noncomparable. Then A is a trellis.

Example 2.2. let A be a set $\{0; 1; a; b; c; d\}$ with the following pseudo-order: $a E c E d, b E d, b E c E d, 0 E x E 1$ for every $x \in \{a; b; c; d\}$ and $0 E 1$. Then A is a trellis but not lattice and associative since, $(a \vee b) \vee d = d$ but $a \vee (b \vee d) = 1$:

Some properties on lattices hold in trellises as following:

- $p_1) x \wedge y = y \wedge x; x \vee y = y \vee x; (\text{commutativity})$
- $p_2) (x \wedge y) \vee x = x; (x \vee y) \wedge x = x; (\text{absorption})$
- $p_3) x \vee ((x \wedge y) \vee (x \wedge z)) = x \wedge ((x \vee y) \wedge (x \vee z)); (\text{part - preservation})$

Theorem 2.3. Let $(T; \sqcup)$ be a trellis. Then by taking $x \sqcap y = g:l;b\{x; y\}$ and $x \sqcup y = l:u;b\{x; y\}$ the binary operation \sqcup and \sqcap satisfy in $p_1; p_2; p_3$:

From now on, by trellis $(T; \sqcup; \sqcap)$ we mean $(T; \sqcup)$ that $x \sqcup y$ is defined by $x \sqcap y = x$ or $x \sqcup y = y$:

Remark 2.4. It is trivial that every associative trellis is a lattice.

Theorem 2.5. A set T with two commutative, absorption and part-preserving operations " \sqcup ", " \sqcap " is a trellis if $a \sqcup b$ is defined as $a \sqcap b = a$ or $a \sqcup b = b$.

Proof. Refer to [3, page on 219].

Definition.

(i) A subtrellis S of a trellis $(T; \sqcup; \sqcap)$ is a nonempty subset of T such that $a; b \in S$ implies $a \sqcap b$ and $a \sqcup b$ belong to S .

(ii) An ideal I of a trellis T is a subtrellis of T such that $i \in I$ and $a \in T$ imply that $a \sqcap i \in I$ or equivalently for any $i \in I$ and $a \in T$, $a \sqcup i$ implies $a \in I$. Moreover, an ideal I of a trellis T is called a prime ideal if $x \sqcap y \in I$ implies $x \in I$ or $y \in I$ for all $x; y \in T$. Note that if $I_1; I_2$ are ideals of a trellis T , so is $I_1 \cap I_2$:

(iii) A trellis T is *modular* if the following condition holds for all $x; y; z \in T$; $x \sqcup z \Rightarrow x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$:

Theorem 2.6. In any trellis, the following statements are equivalent:

- \sqcup is transitive,
- the operation \sqcup and \sqcap are associative,
- one of the operations \sqcup or \sqcap is associative.

Proof. Refer to [3, Theorem 3].

As [2, lemma 2.2] by simple argument we have the following lemma:

Lemma 2.7. Let $k : T \rightarrow T$ be a mapping on a trellis T satisfying the property

$k(x \sqcap y) = kx \sqcap y$. Then for all $x; y \in T$:

- $x \sqcup y$ implies $kx \sqcup y$;
- $kx \sqcup x$;
- $k(kx) = kx$, i.e. k is idempotent;
- $k(x \sqcap y) = kx \sqcap ky$;

v) the *fixed* elements of k (x is said to be a *fixed* element of k if $kx = x$)

form an ideal of T which will be called the *fixed* ideal of k , denoted by $\text{Fix } k$, also

$\text{Fix } k = k(T)$.

Proposition 2.8. If A is an ideal of a trellis T and k is a mapping $k : T \rightarrow T$ satisfying $k(x \sqcap y) = kx \sqcap y$, then $k(A)$ is an ideal of A and hence an ideal of T :

Proof. Refer to [2, proposition 2.6].

Although some trellis are not distributive, by a weaker condition we have the following:

Proposition 2.9. If a trellis T satisfies the inequality $x \sqcap (y \sqcup z) \sqcup (x \sqcap y) \sqcup (x \sqcap z)$; then every mapping k on T satisfying $k(x \sqcap y) = kx \sqcap y$, implies $k(x \sqcup y) = kx \sqcup ky$:

Proof. Suppose that k be a mapping on T satisfying the property above. For any $x; y \in T$,
 $k(x \sqcup y) = k(x \sqcup y) \sqcap (x \sqcup y)$
 $= E(k(x \sqcup y) \sqcap x) \sqcup (k(x \sqcup y) \sqcap y)$
 $= k((x \sqcup y) \sqcap x) \sqcup k((x \sqcup y) \sqcap y)$
 $= kx \sqcup ky$:

Since $x; y \sqcup x \sqcup y$ we have $kx; ky \sqcup k(x \sqcup y)$; and so $kx \sqcup ky \sqcup k(x \sqcup y)$:

Therefore $k(x \sqcup y) = kx \sqcup ky$:

3. On Derivations of Trellises

Definition. Let $T; \sqcap; \sqcup$ be a trellis. A mapping d of a trellis T into itself is called a derivation of T if it satisfies the following condition for all $x; y \in T$:

$d(x \sqcap y) = (d(x) \sqcap y) \sqcup (x \sqcap d(y))$

We can often $d(x)$ written as an abbreviation dx :

Example 3.1.

(i) Let T be a trellis with the least element 0. We define a function d by $dx = 0$ for all $x \in T$. Then d is a derivation on T , which is called the zero derivation.

(ii) Let d be the identity function on a trellis T . Then d is a derivation on T , which is called the identity derivation.

Example 3.2. let $T = \{0; 1; a; b\}$ be a trellis with the following pseudo-order: $a \sqcup b, 0 \sqcup x \sqcup 1$ for every $x \in \{a; b\}$ and $0 \sqcup 1$. We define two functions d_1, d_2 on T by:

$$\begin{array}{lcl}
 d_1x = & \begin{array}{l} \geq x \quad x \equiv 0; 1 \\ < a \quad x = b \\ > \end{array} & \begin{array}{l} 0 \quad x \equiv 0; 1 \\ < a \quad x = b \\ > \end{array} \\
 & : & :
 \end{array}$$

d_2 is a derivation on T but d_1 is not a derivation, because $d_1(a \wedge a) = (d_1a \wedge a)$ implies $b \neq a$:

Example 3.3. let $T = \{0; 1; a; b; c; d\}$ be a trellis with the following pseudo-order: $a \leq c, c \leq d, b \leq d, b \leq c, 0 \leq x \leq 1$ for every $x \in \{a; b; c; d\}$ and $0 \leq 1$. Define $d : T \rightarrow T$ by:

$$\begin{array}{lcl}
 & \begin{array}{l} \geq \\ > 0 \\ < \\ dx = \\ > \end{array} & \begin{array}{l} x = 0; 1 \\ \\ \\ a \quad x = a \\ \end{array}
 \end{array}$$

d is a derivation on T :

Example 3.4. Every mapping k on trellis T satisfying $k(x \wedge y) = kx \wedge y$ is a derivation on T : The above example is a derivation on T that it does not satisfy in this property.

Remark 3.5. It should be noted that principle derivation on lattices defined in

[7] is not a derivation on a trellis. Because it does not have associative property, necessarily.

Proposition 3.6. Let T be a trellis and d be a derivation on T . Then the following statements hold:

- $dx \leq x$
- If I is an ideal of T , then $dI \subseteq I$
- If T has a greatest element 1 and d is a derivation on T , then $dx = (x \wedge d1) \vee dx$ for all $x \in T$:
- If T has a least element 0 and a greatest element 1 , then $d0 = 0$ and $d1 \wedge x \leq dx$:
Proof. (i) If $x \in T$; then $dx = d(x \wedge x) = (dx \wedge x) \vee (x \wedge dx) = x \wedge dx$:
(ii) If I is an ideal of T , then for any $x \in T$; $dx \leq x$ implies that $dx \in I$, thus $dI \subseteq I$:

(iii) Note that $dx = d(x \wedge 1) = (dx \wedge 1) \vee (x \wedge d1) = dx \vee (x \wedge d1)$:

(iv) It is trivial that $d0 = 0$. If $x \in T$; we have,

$$dx = d(x \wedge 1) = (dx \wedge 1) \vee (x \wedge d1) \text{ implies } dx = dx \vee (x \wedge d1)$$

then, $d1 \wedge x \leq dx$:

By applying proposition 3.6 (iii), in the cases $d1 \leq x$ and $x \leq d1$ we have the following:

Corollary 3.7. If T has a greatest element 1 and d is a derivation on T , then for all $x \in T$ we have,

- $d1 \leq x$ implies $d1 \leq dx$:
- $x \leq d1$ implies $dx = x$:

Corollary 3.8. Let T be a trellis with a greatest element 1 and d be a derivation on T : Then $d1 = 1$ if and only if d is the identity derivation.

Remark 3.9. As derivation on trellises for a derivation d satisfying the dual formula of $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$, i.e. $d(x \vee y) = (dx \vee y) \wedge (x \vee dy)$; implies that d is a identity derivation.

Proposition 3.10. Let T be a trellis and d be a derivation on T : Then the following conditions are equivalent:

- d is the identity derivation;
- $d(x \vee y) = (dx \vee y) \wedge (x \vee dy)$;

Proof. The implication (i) \Rightarrow (ii) is trivial.

Taking $x = y$ with together contraction property of d implies (ii) \Rightarrow (i):

Definition. Let T be a trellis and d be a derivation on T . If $x \leq y$ implies $dx \leq dy$, we call d is an isotone derivation.

Example 3.11. The example of 3.2, d_2 is an isotone derivation but in 3.3, d is not an isotone derivation since, $a \leq c$ then $da = a$; $dc = b$ that ab :

Proposition 3.12. Let T be a trellis with a greatest element 1 and d be a derivation on T . If d is an isotone derivation Then $dx = x \wedge d1$:

Proof. Since d is an isotone, then $dx \leq d1$: Note that $dx \leq x$; we can get $dx \leq (x \wedge d1)$, by proposition 3.6(iii), $dx = dx \vee (x \wedge d1) = x \wedge d1$:

Remark 3.13. The above proposition illustrates a condition that makes isotone derivation, principle [7].

Lemma 3.14. Let T be a trellis and $d : T \rightarrow T$ be a derivation. Then $d(dx) = dx$:

Proof. We can get, $dx \in (dx \wedge dx) \vee (d(dx) \wedge x) = d(x \wedge dx) = d(dx)$ and also by 3.6, $d(dx) \in dx$ thus, $d(dx) = dx$:

Theorem 3.15. *Let T be a trellis and $d : T \rightarrow T$ be a derivation satisfying*

$d(x \vee y) = dx \vee dy$: Then for all $x; y \in T$:

- i) *d is a isotone derivation;*
- ii) *$x \in y$ implies $dx = x \wedge dy$;*
- iii) *$dx \wedge y = dx \wedge dy$;*

Proof. (i) Let $x \in y$, then $x \vee y = y$ and so $dx \in (dx \vee dy) = d(x \vee y) = dy$:

(ii) Let $x \in y$: Then by (i), $dx \in dy$ and $dx \in x$: Therefore $dx \in x \wedge dy$: Also $x \wedge dy \in dx$ since, $dx = d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$:

(iii) By definition of the derivation, we have $dx \wedge y \in d(x \wedge y)$ for all $x; y \in L$. Taking $x = dx \wedge y$ and $y = dx$ in (ii), we have $d(dx \wedge y) = (dx \wedge y) \wedge d(dx) = (dx \wedge y) \wedge dx = dx \wedge y$: Thus $(dx \wedge y) \vee (dx \wedge dy) = dx \wedge y$ implies $dx \wedge dy \in dx \wedge y$:

Since $dx \wedge y \in y$ thus $d(dx \wedge y) \in dy$: Then by above equality we have $dx \wedge y \in dy$:

Also $dx \wedge y \in dx$: Then we can get, $dx \wedge y \in dx \wedge dy$: With attention to above

inequalities we have $dx \wedge y = dx \wedge dy$:

Corollary 3.16. *Let T be a trellis and $d : T \rightarrow T$ be a derivation satisfying*

$d(x \vee y) = dx \vee dy$: Then for all $x; y \in T$, $d(x \wedge y) = dx \wedge y$:

Proof. We have $d(x \wedge y) \in dx \wedge dy$; since d is isotone and if $x \wedge y \in x; y$ then $d(x \wedge y) \in dx \wedge dy$. By definition of the derivation, we can get the inverse

relation and so $d(x \wedge y) = dx \wedge y$ for all $x; y \in L$:

Corollary 3.17. *If a trellis T satisfies the inequality $x \wedge (y \vee z) \in (x \wedge y) \vee (x \wedge z)$ and d be a derivation on T , Then the following conditions are equivalent:*

- (i) *$d(x \vee y) = dx \vee dy$;*
- (ii) *$d(x \wedge y) = dx \wedge y$;*

Corollary 3.18. *Let T be a trellis and d be a derivation on T : If $d(x \wedge y) = dx \wedge y$, then $d(x \wedge y) = dx \wedge dy$:*

Proof. We have $d(x \wedge y) = dx \wedge y$ thus, $d(d(x \wedge y)) = d((y \wedge dx))$ then $d(x \wedge y) = dy \wedge dx$:

Corollary 3.19. *Suppose k be a mapping on a trellis T satisfying $k(x \vee y) = kx \vee ky$. Then k is an isotone derivation if and only if $k(x \wedge y) = kx \wedge y$:*

Remark 3.20. This corollary implies that inverse relation 2.7 (i) is established.

Proposition 3.21. *Let T be a trellis and $d : T$*

$d(x \wedge y) = dx \wedge dy$ then d is an isotone derivation.

$\rightarrow T$ be a derivation. If

Proof. For all $x; y$ in T : If $x \leq y$, then $dx = d(x \wedge y) = dx \wedge dy \leq dy$:

Theorem 3.22. Let T be a trellis and d be a derivation on T . Then the following conditions are equivalent:

i) d is an isotone derivation;

ii) $dx \vee dy \leq d(x \vee y)$;

Proof. (i) \Rightarrow (ii). By (i), we have $dx \leq d(x \vee y)$; $dy \leq d(x \vee y)$, and so $dx \vee dy \leq d(x \vee y)$:

(ii) \Rightarrow (i). Assume that (ii) holds. Let $x \leq y$. By (ii), $dx \vee (dx \vee dy) \leq d(y \vee x) = dy$: Thus $dx \leq dy$:

Remark 3.23. Despite lattices, on trellises we can not expect the following statements for an isotone derivation d :

- 1) $d(x \wedge y) = dx \wedge dy$
- 2) $d(x \vee y) = dx \vee dy$
- 3) **Remark 3.24.** It is trivial that every distributive trellis is a modular trellis and every distributive trellis is a associative trellis. Note that by [4, page on 224] every associative trellis is a transitive trellis, and so every distributive trellis is a lattice.

Definition. Let T be a trellis and d be a derivation on T . Define $F_{ix_d}(T) = \{x \in T \mid dx = x\}$: By the following proposition we can see that $F_{ix_d}(T)$ is down-closed set, that is, $x \in F_{ix_d}(T)$ and $y \leq x$ imply $y \in F_{ix_d}(T)$. Moreover if d is isotone, $F_{ix_d}(T)$ is an ideal of T :

Proposition 3.25. Let T be a trellis and d be a derivation on T . If $y \leq x$ and $dx = x$, then $dy = y$.

Proof. suppose $x; y$ are arbitrary elements in L , $y \leq x$, then $y = x \wedge y$: Thus,

$$\begin{aligned} dy &= d(x \wedge y) \\ &= (dx \wedge y) \vee (x \wedge dy) \\ &= (x \wedge y) \vee dy \\ &= y \vee dy \\ &= y: \end{aligned}$$

Theorem 3.26. Let T be a trellis and d_1 and d_2 be two isotone derivations on T . Then $d_1 = d_2$ if and only if $F_{ix_{d_1}}(T) = F_{ix_{d_2}}(T)$:

Proof. Trivially, $d_1 = d_2$ implies $F_{ix_{d_1}}(T) = F_{ix_{d_2}}(T)$: For the converse, for all $x \in T$ since $d_1x \in F_{ix_{d_1}}(T) = F_{ix_{d_2}}(T)$ we have $d_2d_1x = d_1x$: Similarly $d_1d_2x = d_2x$: On the other hand, isotone of d_1 and d_2 implies that $d_2d_1x \leq d_2x = d_1d_2$ and $d_2d_1x \leq d_1d_2x$: Also, $d_1d_2x \leq d_2d_1x$; this show that $d_2d_1x = d_1d_2x$: It follows that $d_1x = d_2d_1x = d_1d_2x = d_2x$:

Definition. Let $(A_1; E_1)$ and $(A_2; E_2)$ two pseudo-ordered set. By $(A_1 \times A_2; E)$ we means the set $A_1 \times A_2$ with the pseudo-order $(a_1; a_2) \leq (b_1; b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$: If T_1 and T_2 are trellises, so is $T_1 \times T_2$.

Remark 3.27. $T_1; A_1; V_1, T_2; A_2; V_2$ are trellises. It consider that for all $a_1; b_1 \in T_1$ and $a_2; b_2 \in T_2$, $(T_1 \times T_2; V; \wedge)$ with $(a_1; a_2) \wedge (b_1; b_2) = (a_1 \wedge b_1; a_2 \wedge b_2)$ and $(a_1; a_2) \vee (b_1; b_2) = (a_1 \vee b_1; a_2 \vee b_2)$ is a trellis.

Definition. Suppose $d_1; d_2$ are arbitrary derivations on $T_1; T_2$ respectively. Define $d : T_1 \times T_2 \rightarrow T_1 \times T_2$: $d(a; b) = (d_1a; d_2b)$ for all $a \in T_1; b \in T_2$: Trivially, d is a derivation and it is called a *Cartesian derivation*.

Example 3.28. Cartesian derivation of identity derivations is the identity derivation and if $d_1; d_2$ are isotone derivations then $d = d_1 \times d_2$ is an isotone derivation.

Remark 3.29. Nonetheless, we can product many derivations in such matter, but there is a derivation on $T_1 \times T_2$ that is not a cartesian derivation. For, let $T_1 = T_2 = T = \{0; 1\}$ be a trellis with $0 \leq 1$ and define $d : T \times T \rightarrow T \times T$ by $d(x; y) = (x; y)$ for all $(x; y) \neq (1; 1)$ and $d(1; 1) = (1; 0)$: Note that this derivation is not isotone, perhaps isotone derivation on $T_1 \times T_2$ are cartesian derivations.

Definition. Let T be a trellis. A function $d : T \rightarrow T$ is called an *f-derivation*

on T if there exists a function $f : T \rightarrow T$ such that

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y));$$

for all $x; y \in T$:

Remark 3.30. It is obvious that if f is an identity function then d is a derivation on T .

Example 3.31. let $T = \{0; 1; a; b; c\}$ be a trellis with the following pseudo-order: $a \leq b \leq c$, $0 \leq x \leq 1$ for every $x \in \{a; b; c\}$ and $0 \leq 1$. Define $d : T \rightarrow T$ by:

$$\begin{aligned} d0 &= 0; & d1 &= 1 \\ d a &= a; & d b &= b; & d c &= c. \end{aligned}$$

Then d is not a derivation on T since $0 = d(a \wedge 1) \neq (da \wedge 1) \vee (a \wedge d1) = 0 \vee a = a$: If we define f by :

$$\begin{aligned} f0 &= 0; & f1 &= 1 \\ f a &= a; & f b &= b; & f c &= c. \end{aligned}$$

then d is an f -derivation on T , for all $x, y \in T$.

Proposition 3.32. Let T be a trellis and d be a f -derivation on T . Then the following identities hold for all $x, y \in T$

- (i) $dx \leq fx$
- (ii) If T has a least element 0 ; then $f0 = 0$ implies $d0 = 0$:

Proof. (i) For all $x \in T$; we have

$$dx = d(x \wedge x) = dx \wedge fx; \text{ thus } dx \leq fx:$$

- (ii) Since $dx \leq fx$ for all $x \in T$, we have $0 \leq d0 \leq f0 = 0$:

Corollary 3.33. If T has a greatest element 1 and d is an f -derivation on T , $f1 = 1$, then for all $x \in T$ we have,

- (i) If $fx \leq d1$, then $dx = fx$;
- (ii) If $d1 \leq fx$, then $d1 \leq dx$:

Proof. (i) we have $dx = d(x \wedge 1) = (dx \wedge f1) \vee (fx \wedge d1) = dx \vee fx$, then $fx \leq dx$: From proposition 3.32 (i), we obtain $dx = fx$:

- (ii) Since $dx = d(x \wedge 1) = (dx \wedge f1) \vee (fx \wedge d1) = dx \vee d1$, we have $d1 \leq dx$:

Remark 3.34. Note that if $d1 = 1$; since $d1 \leq f1$; we have $f1 = 1$. In this case from corollary 3.33 (i), we get $dx = fx$:

Definition. Let T be a trellis and d be an f -derivation on T . If $x \leq y$ implies $dx \leq dy$, we call d is an *isotone f -derivation*.

Example 3.35. The example of 3.31, d is not an isotone f -derivation, since $c \leq 1$ but it does not

follow $dc \leq d1$; whereas f is an increasing function on T :

Corollary 3.36. Let T be a trellis and d be an f -derivation on T . Then for all $x, y \in T$ we have,

- (i) If d is an isotone f -derivation, then $dx \vee dy \leq d(x \vee y)$;
- (ii) If $d(x \wedge y) = dx \wedge dy$, then d is an isotone f -derivation.

Proof. (i) We know that $x \leq x \vee y$ and $y \leq x \vee y$: Since d is isotone, $dx \leq d(x \vee y)$ and $dy \leq d(x \vee y)$: Hence we obtain $dx \vee dy \leq d(x \vee y)$:

- (ii) Let $d(x \wedge y) = dx \wedge dy$ and $x \leq y$: Since $dx = d(x \wedge y) = dx \wedge dy$; we get $dx \leq dy$:

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