

The Extended Exponential Distribution and Its Applications

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We introduce a new three-parameter extension of the exponential distribution called the *odd exponentiated half-logistic exponential* distribution. Various of its properties including quantile and generating functions, ordinary and incomplete moments, mean residual life, mean inactivity time and some characterizations are investigated. The new density function can be expressed as a linear mixture of exponential densities. The hazard rate function of the new model can be decreasing, increasing or bathtub shaped. The maximum likelihood is used for estimating the model parameters. We prove empirically the flexibility of the new distribution using two real data sets.

Keywords: Exponential distribution; Characterization; Generating function; Maximum likelihood, Simulation.

1. Introduction

Recently, there are hundreds of generalized distributions which have several applications from biomedical sciences, environmental, engineering, financial, among other fields. These applications have indicated that many data sets following the classical distributions are more often the exception rather than the reality. Hence, the statisticians have been made a significant progress towards the generalization of the classical models and their successful applications in applied areas.

Many authors developed generalizations of the exponential (Ex) distribution. For example, Gupta and Kundu [4] defined the exponentiated Ex (EEx), Nadarajah and Kotz [8] proposed the beta Ex (BEx), Cordeiro et al. [3] pioneered the Kumaraswamy Ex (KEx) as a special case form the Kumaraswamy Weibull distribution, Khan et al. [5] introduced the transmuted generalised Ex (TGEx), Mahdavi and Kundu [6] studied the alpha power Ex (APEX) and Afify et al. [2] studied the Kumaraswamy transmuted Ex (KTEEx) distributions.

Afify et al. [1] proposed a wider class of distributions called the odd exponentiated half-logistic- G (OEHL- G) family. The cumulative distribution function (CDF) and probability density function (PDF) of the OEHL- G family are given by

$$F(x; \alpha, \theta, \xi) = \left\{ \frac{1 - \exp \left[\frac{-\theta G(x; \xi)}{\tilde{G}(x; \xi)} \right]}{1 + \exp \left[\frac{-\theta G(x; \xi)}{\tilde{G}(x; \xi)} \right]} \right\}^{\alpha} \quad (1.1)$$

and

$$f(x; \alpha, \theta, \xi) = \frac{2\alpha\theta g(x, \xi) \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right] \left\{1 - \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\}^{\alpha-1}}{\bar{G}(x, \xi)^2 \left\{1 + \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\}^{\alpha+1}}, \quad (1.2)$$

respectively, where $g(x; \xi) = dG(x; \xi)/dx$, α and θ are positive shape parameters and ξ denotes the vector of parameters for baseline CDF G . Henceforth, a random variable (rv) with density (1.2) is denoted by $X \sim \text{OEHL-}G(\alpha, \theta, \xi)$.

The hazard rate function (HRF) of X is given by

$$h(x; \alpha, \theta, \xi) = \frac{2\alpha\theta \frac{\tau(x, \xi)}{\bar{G}(x, \xi)} \left\{1 + \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\}^{-1} \left\{1 - \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\}^{\alpha-1}}{\exp\left[\frac{\theta G(x, \xi)}{\bar{G}(x, \xi)}\right] \left(\left\{1 + \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\}^{\alpha} - \left\{1 - \exp\left[\frac{-\theta G(x, \xi)}{\bar{G}(x, \xi)}\right]\right\} \right)}.$$

Further information about the OEHL- G can be explored in Afify et al. [1].

In this paper, we propose and study a new three-parameter model called the *odd exponentiated half-logistic exponential* (OEHLEx) distribution. Based on the OEHL- G family proposed by Afify et al. [1], we construct the new OEHLEx model and give a comprehensive description of some of its mathematical properties. The OEHLEx distribution has some desirable properties like its capability to model monotone and bathtub hazard rates. We show, using two applications, that the OEHLEx distribution can provide better fits than 8 other well-known competing extensions of the Ex distribution.

The rest of the paper is organized as follows. In Section 2, we define the OEHLEx distribution, provide a useful linear representation for its PDF and give some plots for its PDF and HRF. We obtain, in Section 3, some mathematical properties of the proposed distribution including quantile and generating functions, ordinary and incomplete moments, mean residual life, mean inactivity time and some characterization results. In Section 4, we obtain the maximum likelihood estimators (MLEs) of the model parameters and assess the performance of these estimates via a Monte Carlo simulation study. In Section 5, we illustrate the importance of the OEHLEx model using two applications to real data. Finally, in Section 6, we provide some concluding remarks.

2. The OEHLEx distribution

In this section, we define the OEHLEx distribution and provide some plots for its PDF and HRF. By taking $G(x; \xi)$ and $g(x; \xi)$ in (1.2) to be the CDF and PDF of the Ex distribution with PDF and CDF

$$g(x; \lambda) = \lambda \exp(-\lambda x) \quad \text{and} \quad G(x; \lambda) = 1 - \exp(-\lambda x), \quad (2.1)$$

respectively, where $\lambda > 0$ is a scale parameter.

Then, the PDF of the OEHLEx is given (for $x > 0$) by

$$f(x) = \frac{2\alpha\theta\lambda \exp\left\{\frac{-\theta[1-\exp(-\lambda x)]}{\exp(-\lambda x)}\right\} \left(1 - \exp\left\{\frac{-\theta[1-\exp(-\lambda x)]}{\exp(-\lambda x)}\right\}\right)^{\alpha-1}}{\exp(-\lambda x) \left(1 + \exp\left\{\frac{-\theta[1-\exp(-\lambda x)]}{\exp(-\lambda x)}\right\}\right)^{\alpha+1}}, \quad (2.2)$$

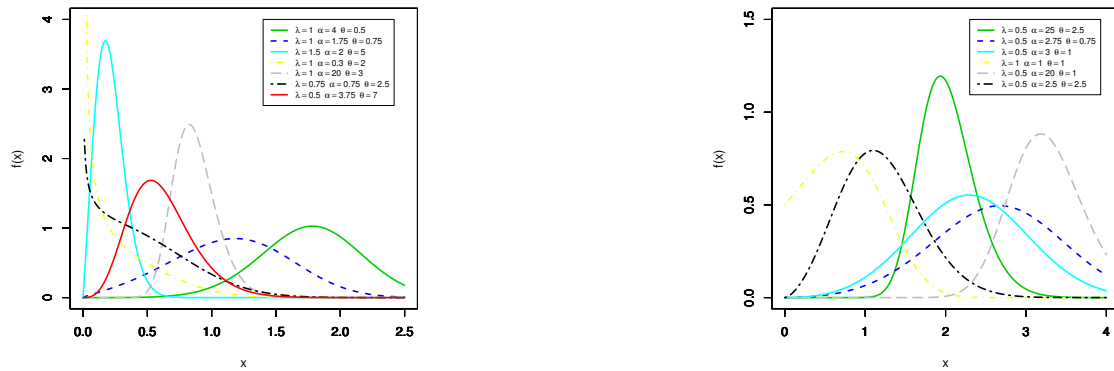


Fig. 1. PDF plots of the OEHLEx model for selected parameters values.

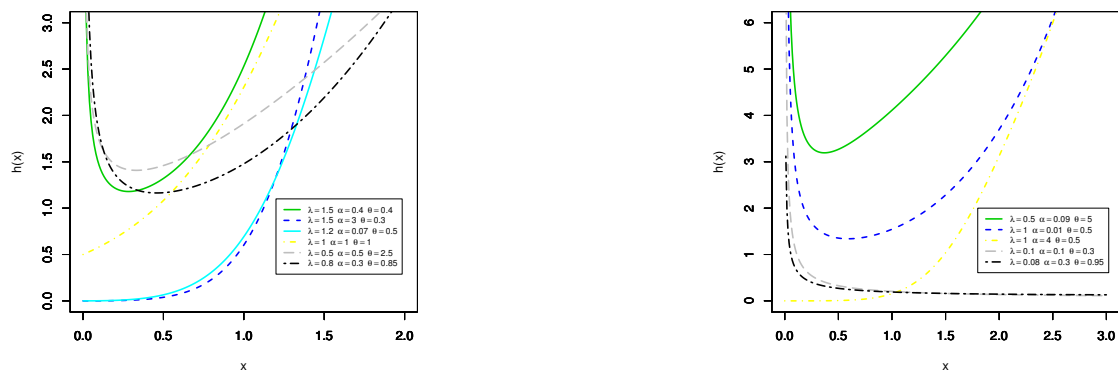


Fig. 2. HRF plots of the OEHLEx model for selected parameters values.

where α and θ are positive shape parameters and λ is a positive scale parameter. Hereafter, a rv with PDF (2.2) is denoted by $X \sim \text{OEHLEx}(\lambda, \alpha, \theta)$.

The CDF of the OEHLEx distribution reduces to

$$F(x) = \left(\frac{1 - \exp \left\{ \frac{-\theta[1 - \exp(-\lambda x)]}{\exp(-\lambda x)} \right\}}{1 + \exp \left\{ \frac{-\theta[1 - \exp(-\lambda x)]}{\exp(-\lambda x)} \right\}} \right)^{\alpha}. \quad (2.3)$$

Plots of the OEHLEx PDF and HRF for selected parameter values are shown in Figures 1 and 2, respectively. The PDF of the OEHLEx distribution can be symmetric, reversed-J shaped, right skewed or left skewed. The HRF of the OEHLEx distribution can be decreasing, increasing or bathtub shaped.

Remark 1: The PDF of the OEHLEx distribution can be expressed as a mixture of Ex densities

$$f(x) = \sum_{m=0}^{\infty} \omega_m g_{m+1}(x; (m+1)\lambda), \quad (2.4)$$

where $g_{m+1}(x; (m+1)\lambda) = (m+1)\lambda \exp[-(m+1)\lambda x]$ is the Ex PDF with scale parameter $(m+1)\lambda$ and

$$\begin{aligned} \omega_m = 2\alpha\theta \sum_{k,l,j,i=0}^{\infty} \frac{(-1)^{j+k+l+m} [\theta(j+i+1)]^k}{k!(m+1)} \\ \times \binom{-\alpha-1}{i} \binom{\alpha-1}{j} \binom{-k-2}{l} \binom{k+l}{m}. \end{aligned}$$

Based on Remark 1, several mathematical properties of the OEHLEx distribution can be obtained simply from those properties of the Ex distribution.

Proof.

According to Equation (8) in Afify et al. [1], the PDF of the OEHLEx distribution can be expressed as

$$f(x) = \sum_{k,l=0}^{\infty} a_{k,l} h_{k+l+1}(x), \quad (2.5)$$

where $h_{k+l+1}(x) = (k+l+1)\lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{k+l}$ is the exponentiated Ex density with power parameter $(k+l+1)$ and

$$a_{k,l} = 2\alpha\theta \sum_{j,i=0}^{\infty} \frac{(-1)^{j+k+l} [\theta(j+i+1)]^k}{k!(k+l+1)} \binom{-\alpha-1}{i} \binom{\alpha-1}{j} \binom{-k-2}{l}.$$

Applying the generalized binomial series to (2.5), we obtain

$$f(x) = \sum_{m=0}^{\infty} \omega_m g_{m+1}(x; (m+1)\lambda).$$

3. Properties and characterization

In this section, we provide some properties of the OEHLEx distribution including quantile and generating functions, ordinary and incomplete moments, mean residual life, mean inactivity time and some characterizations.

Henceforth, let Y be a rv having the Ex distribution in (2.1). Thus, the r th ordinary, incomplete moments and moment generating function (MGF) of Y are given by

$$\mu'_{r,Y} = \lambda^{-r} \Gamma(r+1), \phi_{r,Y}(t) = \lambda^{-r} \gamma(r+1, \lambda t) \text{ and } M_Y(t) = \frac{\lambda}{\lambda - t}, t \neq 0,$$

respectively, where $\Gamma(\cdot)$ and $\gamma(\cdot, \cdot)$ are, respectively, the complete and lower incomplete gamma functions.

3.1. Quantile and generating functions

The quantile function (QF) of the OEHLEx distribution follows, by inverting (2.3), as

$$x_u = Q(u) = \frac{-1}{\lambda} \log \left[1 - \frac{-\log(1 - u^{\frac{1}{\alpha}}) + \log(1 + u^{\frac{1}{\alpha}})}{\theta - \log(1 - u^{\frac{1}{\alpha}}) + \log(1 + u^{\frac{1}{\alpha}})} \right]. \quad (3.1)$$

Based on (3.1), we can obtain a random sample of size n from (2.3), as $X_i = Q(U_i)$, where $U_i \sim \text{Uniform}(0, 1)$, $i = 1, 2, \dots, n$.

Using (2.4) and the MGF of Y , we can write the MGF of the OEHLEx model as

$$M_X(t) = \sum_{m=0}^{\infty} \omega_m \frac{(m+1)\lambda}{(m+1)\lambda - t}.$$

3.2. Ordinary and incomplete moments

The s th ordinary moment of X follows from (2.4) as

$$\mu'_s = \sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^{-s} \Gamma(s+1).$$

Using (2.4), the s th incomplete moment of X can be determined as

$$\phi_s(t) = \sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^{-s} \gamma(s+1, (m+1)\lambda t).$$

The first incomplete moment follows from the above equation with $s = 1$ as

$$\phi_1(t) = \sum_{m=0}^{\infty} \omega_m \frac{\gamma(2, (m+1)\lambda t)}{[(m+1)\lambda]}. \quad (3.2)$$

3.3. Mean residual life and mean inactivity time

The life expectancy at age t (for $t > 0$) or mean residual life (MRL) of X is defined by

$$m_X(t) = [1 - \phi_1(t)] / \bar{F}(t) - t, \quad (3.3)$$

where $\bar{F}(\cdot)$ is the survival function.

By replacing (3.2) in Equation (3.3), the MRL of X can be determined as

$$m_X(t) = \frac{1}{\bar{F}(t)} \sum_{m=0}^{\infty} \omega_m \frac{\gamma(2, (m+1)\lambda t)}{(m+1)\lambda} - t.$$

The mean inactivity time (MIT) of X is defined (for $t > 0$) by

$$m'_X(t) = t - [\phi_1(t) / F(t)].$$

By inserting (3.2) in the last equation, the MIT of X follows as

$$m'_X(t) = t - \frac{1}{F(t)} \sum_{m=0}^{\infty} \omega_m \frac{\gamma(2, (m+1)\lambda t)}{(m+1)\lambda}.$$

3.4. Characterizations

We will use the following two lemmas to prove the characterization of the OEHLEx distribution.

Assumption A

Suppose that X is an absolutely continuous rv with CDF $F(x)$ and PDF $f(x)$. We assume $E(X)$ exists and $f(x)$ is differentiable. We assume further

$$\alpha = \sup\{x | f(x) > 0\} \text{ and } \beta = \inf\{x | f(x) < 1\}.$$

Lemma 1.

If $E(X|X \leq x) = g(x) \frac{f(x)}{F(x)}$, where $g(x)$ is a continuous differentiable function in (α, β) , then $f(x) = ce^{\int \frac{x-g'(x)}{g(x)} dx}$, c is determined by the condition $\int_{\alpha}^{\beta} f(x) dx = 1$.

Proof.

$$g(x) = \frac{\int_{\alpha}^x uf(u) du}{f(x)}.$$

Thus

$$\int_{\alpha}^x uf(u) du = f(x)g(x).$$

Differentiating both sides of the above equation, we obtain

$$xf(x) = f'(x)g(x) + f(x)g'(x),$$

on simplification, we get

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

By integrating both sides of the above equation, we obtain

$$f(x) = ce^{\int \frac{x-g'(x)}{g(x)} dx}, c \text{ is determined by the condition } \int_{\alpha}^{\beta} f(x) dx = 1.$$

Lemma 2.

If $E(X|X \geq x) = h(x) \frac{f(x)}{1-F(x)}$, where $h(x)$ is a continuous differentiable function in (α, β) , then $f(x) = ce^{\int \frac{x+h'(x)}{h(x)} dx}$, c is determined by the condition $\int_{\alpha}^{\beta} f(x) dx = 1$.

Proof.

$$h(x) = \frac{\int_x^{\infty} uf(u) du}{f(x)}.$$

Thus

$$\int_x^\infty u f(u) du = f(x) g(x).$$

Differentiating both sides of the above equation, we obtain

$$-x f(x) = f'(x) h(x) + f(x) h'(x),$$

on simplification, we get

$$\frac{f'(x)}{f(x)} = -\frac{x + h'(x)}{h(x)}.$$

By integrating both sides of the above equation, we obtain

$$f(x) = c e^{\int -\frac{x+h'(x)}{h(x)} dx}, \quad c \text{ is determined by the condition } \int_\alpha^\beta f(x) dx = 1.$$

Theorem 1.

Suppose that X is an absolutely continuous rv with CDF $F(x)$ and PDF $f(x)$. We assume $E(X)$ exists and $f(x)$ is differentiable. We assume further

$$0 = \sup\{x | f(x) > 0\} \quad \text{and} \quad \infty = \inf\{x | f(x) < 1\}.$$

Then

$$E(X | X \leq x) = g(x) \frac{f(x)}{F(x)},$$

where

$$g(x) = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda] \gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda e^{-(m+1)\lambda x}}, \quad \gamma(2, s) = \int_0^s u e^{-u} du$$

and

$$\begin{aligned} \omega_m &= 2\alpha\theta \sum_{k,l,j,i=0}^{\infty} \frac{(-1)^{j+k+l+m} [\theta(j+i+1)]^k}{k! (m+1)} \\ &\quad \times \binom{-\alpha-1}{i} \binom{\alpha-1}{j} \binom{-k-2}{l} \binom{k+l}{m}. \end{aligned}$$

If and only if

$$f(x) = \sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x], \quad \lambda > 0.$$

Proof.

We have

$$\begin{aligned} g(x)f(x) &= \sum_{m=0}^{\infty} \omega_m \int_0^x (m+1)\lambda u \exp[-(m+1)\lambda u] du \\ &= \sum_{m=0}^{\infty} \omega_m [(m+1)\lambda] \gamma(2, (m+1)\lambda x). \end{aligned}$$

Therefore

$$g(x) = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda] \gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}.$$

Suppose

$$g(x) = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda] \gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]},$$

then

$$\begin{aligned} g'(x) &= x - \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda] \gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]} \times \\ &\quad \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]} \\ &= x - g(x) \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 e^{-(m+1)\lambda x}}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}. \end{aligned}$$

Thus

$$\frac{x - g'(x)}{g(x)} = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}.$$

By Lemma 1, we have

$$\frac{f'(x)}{f(x)} = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}}.$$

On integrating the above expression, we obtain

$$f(x) = c \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]},$$

where c is a constant. Using the condition, $\int_0^{\infty} f(x) = 1$, we obtain

$$f(x) = \sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x].$$

Theorem 2.

Suppose that X is an absolutely continuous rv with CDF $F(x)$ and PDF $f(x)$. We assume $E(X)$ exists and $f(x)$ is differentiable. We assume further

$$0 = \sup\{x | f(x) > 0\} \quad \text{and} \quad \infty = \inf\{x | f(x) < 1\}.$$

Then

$$E(X|X \geq x) = h(x) \frac{f(x)}{F(x)},$$

where

$$h(x) = \frac{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \Gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]}, \quad \Gamma(2, s) = \int_s^{\infty} u e^{-u} du$$

and ω_m is defined before in Theorem 1. If and only if

$$f(x) = \sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x], \quad \lambda > 0.$$

Proof.

We have

$$\begin{aligned} h(x)f(x) &= \sum_{m=0}^{\infty} \omega_m \int_x^{\infty} (m+1) \lambda u \exp[-(m+1)\lambda u] du \\ &= \sum_{m=0}^{\infty} \omega_m (m+1) \lambda \Gamma(2, (m+1)\lambda x). \end{aligned}$$

Therefore

$$h(x) = \frac{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \Gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]}.$$

Suppose

$$h(x) = \frac{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \Gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]},$$

then

$$\begin{aligned} h'(x) &= -x - \frac{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \Gamma(2, (m+1)\lambda x)}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]} \times \\ &\quad \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]} \\ &= -x - h(x) \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1) \lambda \exp[-(m+1)\lambda x]}. \end{aligned}$$

Thus

$$\frac{x + h'(x)}{(x)} = \frac{-\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}.$$

By Lemma 2, we have

$$\frac{f'(x)}{f(x)} = \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]}}.$$

On integrating the above expression, we obtain

$$f(x) = c \frac{\sum_{m=0}^{\infty} \omega_m [(m+1)\lambda]^2 \exp[-(m+1)\lambda x]}{\sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x]},$$

where c is a constant. Using the condition, $\int_0^{\infty} f(x) = 1$, we obtain

$$f(x) = \sum_{m=0}^{\infty} \omega_m (m+1)\lambda \exp[-(m+1)\lambda x].$$

4. Estimation and simulation

Let X_1, \dots, X_n be a random sample from the OEHLEx distribution with parameters λ , α and θ . Let $\varphi = (\lambda, \alpha, \theta)^T$ be the $p \times 1$ parameter vector. Then, the log-likelihood function for φ is given by

$$\begin{aligned} \ell(\varphi) = & n \log(2\alpha) + n \log(\lambda) + n \log(\theta) + \lambda \sum_{i=1}^n x_i - \theta \sum_{i=1}^n \frac{[1 - \exp(-\lambda x_i)]}{\exp(-\lambda x_i)} \\ & + (\alpha - 1) \sum_{i=1}^n \log \left(1 - \exp \left\{ \frac{-\theta [1 - \exp(-\lambda x_i)]}{\exp(-\lambda x_i)} \right\} \right) \\ & - (\alpha + 1) \sum_{i=1}^n \log \left(1 + \exp \left\{ \frac{-\theta [1 - \exp(-\lambda x_i)]}{\exp(-\lambda x_i)} \right\} \right). \end{aligned}$$

The score vector components, $\mathbf{U}(\varphi) = \frac{\partial \ell}{\partial \varphi} = (U_\lambda, U_\alpha, U_\theta)^T$, are available with the corresponding author upon request.

Setting the nonlinear system of equations $U_\lambda = U_\alpha = U_\theta = 0$ and solving them simultaneously yields the MLE $\hat{\varphi}$. This can be adopted using nonlinear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically.

Now, we conduct a Monte Carlo simulation study to assess the performance of the MLEs of the unknown parameters for the OEHLEx distribution. The performance of the MLEs is evaluated in terms of their average values and mean squared errors (MSEs). The Mathcad program is used to generate 1000 samples of the OEHLEx distribution for different sample sizes, where $n = (20, 50, 100, 150)$, and for different parameters combinations, where $\lambda = (1, 1.5)$, $\alpha = (0.5, 0.75, 1.5, 1.75, 3.5)$ and $\theta = (0.35, 0.5, 0.75, 1.5, 3.5)$. The average values of estimates and MSEs are provided in Table 1. It is noted, from Table 1, that the MSE decreases as the sample size increases. Thus, the MLE method works very well to estimate the model parameters of the OEHLEx distribution.

5. Data analysis

In this section, we illustrate the importance of the OEHLEx distribution using two applications to real data. The fits of the OEHLEx distribution will be compared with some competitive models namely: the exponentiated Weibull (EW) Mudholka et al. [7], KEx, KTEEx, BEx, gamma (Ga), TGEEx, EEEx, APEEx and Ex distributions, whose PDFs (for $x > 0$) are given by

$$\text{EW: } f(x) = \beta \lambda \theta x^{\beta-1} \exp(-\lambda x^\beta) [1 - \exp(-\lambda x^\beta)]^{\theta-1}.$$

$$\text{KEx: } f(x) = ab\lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{a-1} \{1 - [1 - \exp(-\lambda x)]^a\}^{b-1}.$$

$$\text{KTEEx: } f(x) = \frac{ab\lambda \exp(-\lambda x) [1 - \theta + 2\theta \exp(-\lambda x)]}{\{[1 - \exp(-\lambda x)][1 + \theta \exp(-\lambda x)]\}^{1-a}} (1 - \{[1 - \exp(-\lambda x)][1 + \theta \exp(-\lambda x)]\}^a)^{b-1}.$$

$$\text{BEx: } f(x) = \frac{\lambda}{B(a,b)} \exp(-b\lambda x) [1 - \exp(-\lambda x)]^{a-1}.$$

$$\text{Ga: } f(x) = \frac{b^{-a}}{\Gamma(a)} x^{a-1} \exp(-x/b).$$

$$\text{TGEEx: } f(x) = \alpha \lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1} \times \{1 + \theta - 2\theta [1 - \exp(-\lambda x)]^\alpha\}.$$

$$\text{EEEx: } f(x) = \alpha \lambda \exp(-\lambda x) [1 - \exp(-\lambda x)]^{\alpha-1}.$$

$$\text{APEEx: } f(x) = \frac{\log(\alpha) \lambda \exp(-\lambda x)}{(\alpha-1)} \alpha^{1-\exp(-\lambda x)}, \alpha > 0, \alpha \neq 1.$$

The parameters of the above densities are all positive real numbers except for the KTEEx and TGEEx distributions for which $|\theta| \leq 1$.

The competitive models are compared using some goodness-of-fit criteria including Cramér-Von Mises (W^*), Anderson-Darling (A^*) and Kolmogorov Smirnov (KS) statistic and its p-value.

The first data set consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The data are reported in Smith et al. [10]. The second data set consists of 100 observations of breaking stress of carbon fibres (in Gba) as reported in Nichols and Padgett [9]

Tables 2 and 3 list the W^* , A^* , KS, its p-value, MLEs and their standard errors (SEs) (in parentheses) for the fitted OEHLEx distribution and other competitive models.

The figures in these tables show that the OEHLEx distribution has the lowest values for all goodness-of-fit statistics among all fitted distributions.

The histogram of the two data sets, and the estimated CDF, SF and PP plots for the OEHLEx distribution are displayed in Figures 3 and 4, respectively. The plots in Figures 3 and 4 reveal that the OEHLEx distribution has a close fit to both data sets.

Table 1. Average values of the estimates and the corresponding MSEs for the OEHLEx distribution.

n	Parameters			Average Values			MSEs		
	λ	α	θ	λ	α	θ	λ	α	θ
20	1.0	0.50	0.50	0.910	0.581	0.721	0.041	0.039	0.131
50				1.058	0.517	0.528	0.124	0.012	1.049
100				1.013	0.515	0.462	0.084	0.006	1.018
150				0.966	0.516	0.631	0.044	0.004	0.135
20	1.0	0.50	1.50	1.096	0.548	1.548	0.078	0.027	0.571
50				1.020	0.523	1.535	0.242	0.009	0.546
100				1.028	0.510	1.579	0.133	0.005	0.498
150				1.043	0.503	1.546	0.062	0.003	0.415
20	1.0	0.50	3.50	1.088	0.552	3.519	0.060	0.029	0.804
50				1.125	0.515	3.586	0.278	0.008	2.716
100				1.046	0.508	3.570	0.077	0.003	1.168
150				1.025	0.503	3.538	0.039	0.002	0.731
20	1.0	1.50	0.50	1.102	1.657	0.647	0.160	0.390	0.766
50				1.052	1.592	0.589	0.086	0.280	0.199
100				1.033	1.525	0.522	0.036	0.103	0.069
150				1.019	1.510	0.515	0.021	0.060	0.043
20	1.0	3.50	0.50	1.140	4.099	0.591	0.144	4.993	0.362
50				1.031	3.973	0.578	0.051	2.879	0.148
100				1.032	3.579	0.505	0.022	0.928	0.050
150				1.027	3.515	0.495	0.014	0.473	0.030
20	1.5	0.75	0.35	1.538	0.913	0.661	0.425	0.215	0.901
50				1.562	0.790	0.452	0.216	0.040	0.143
100				1.490	0.787	0.428	0.101	0.020	0.062
150				1.482	0.772	0.403	0.059	0.012	0.034
20	1.5	0.75	0.75	1.786	0.830	0.897	0.886	0.121	1.413
50				1.612	0.776	0.868	0.288	0.030	0.405
100				1.584	0.755	0.761	0.124	0.013	0.137
150				1.558	0.757	0.751	0.066	0.009	0.089
20	1.5	1.75	0.50	1.616	1.954	0.666	0.304	0.551	0.766
50				1.558	1.815	0.563	0.127	0.199	0.140
100				1.514	1.811	0.557	0.081	0.131	0.068
150				1.502	1.785	0.550	0.064	0.110	0.051

Table 2. The W^* , A^* , KS (P-value in parentheses) and estimates (SEs in parentheses) for data set I.

Distribution	W^*	A^*	KS	Estimates (SEs)			
OEHLEx (λ, α, θ)	0.1357	0.7698	0.1228 (0.2982)	2.4615 (0.4798)	1.5777 (0.5047)	0.0346 (0.0334)	
EW (β, λ, θ)	0.1999	1.1118	0.1462 (0.1351)	7.2846 (1.4868)	0.0194 (0.0210)	0.6712 (0.2209)	
KEx (a, b, λ)	0.2687	1.4724	0.1615 (0.0749)	6.8786 (1.1075)	2553.2883 (5216.1254)	0.2374 (0.1228)	
KTEEx (a, b, θ, λ)	0.2719	1.4899	0.1632 (0.0698)	5.4138 (2.2810)	386.4380 (515.1351)	-0.6552 (0.4995)	0.4231 (0.1449)
BEx (a, b, λ)	0.5687	3.1188	0.2164 (0.0055)	17.4493 (3.0818)	134.5357 (243.8003)	0.0812 (0.1388)	
Ga (a, b)	0.5684	3.1174	0.2164 (0.0055)	17.4396 (3.0780)	11.5737 (2.0723)		
TGEx (α, θ, λ)	0.6992	3.8266	0.2123 (0.0068)	31.1539 (11.1417)	-0.6957 (0.1787)	2.9097 (0.2579)	
EEx (α, λ)	0.7862	4.2869	0.2290 (0.0027)	31.3495 (9.5201)	2.6116 (0.2380)		
APEx (α, λ)	0.6417	3.5241	0.2235 (0.0037)	1920499 (23726.92)	2.0711 (0.0989)		
Ex (λ)	0.5702	3.1270	0.4179 (0.0000)	0.6636 (0.0836)			

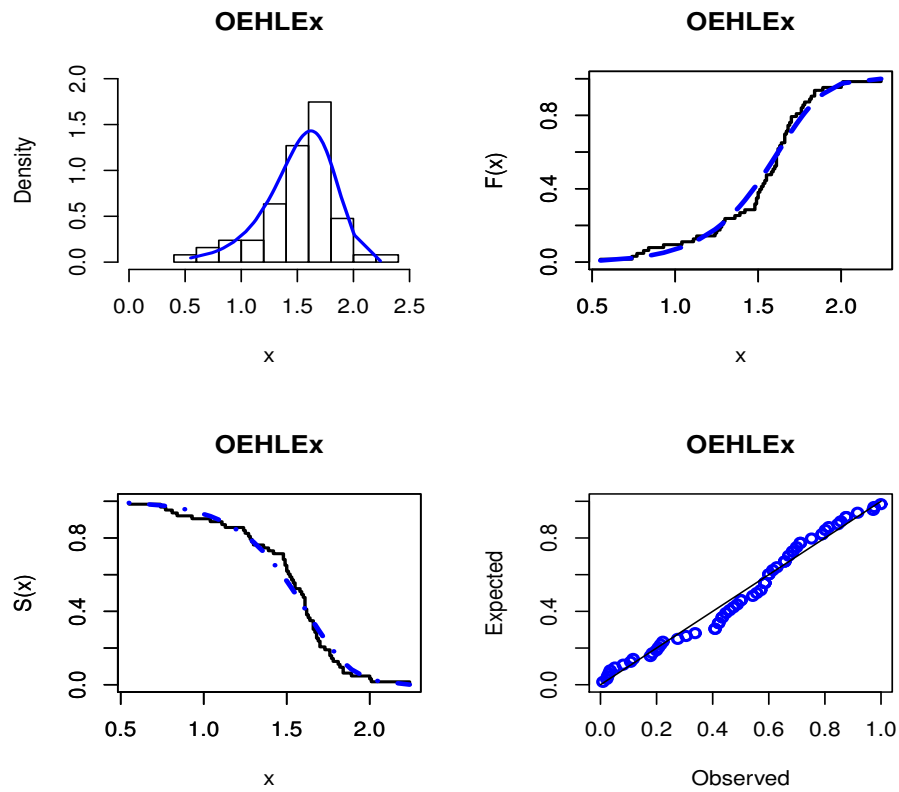


Fig. 3. Fitted PDF, CDF, SF and PP plots of the OEHLEx model for data set I.

Table 3. The W^* , A^* , KS (P-value in parentheses) and estimates (SEs in parentheses) for data set II.

Distribution	W^*	A^*	KS	Estimates (SEs)			
OEHLEx (λ, α, θ)	0.0682	0.4011	0.0639 (0.8088)	0.2669 (0.1042)	2.7131 (0.7322)	2.1000 (1.4127)	
KTEEx (a, b, λ, θ)	0.0685	0.4007	0.0645 (0.7995)	2.6891 (3.2607)	19.4945 (38.547)	0.227 (0.194)	-0.6237 (1.7109)
KEEx (a, b, λ)	0.0699	0.4092	0.0647 (0.7969)	3.2780 (0.8372)	63.2257 (217.67)	0.1134 (0.1823)	
EW (β, λ, θ)	0.0704	0.4132	0.0645 (0.8003)	2.4083 (0.5981)	0.0929 (0.0908)	1.3175 (0.5905)	
BEx (a, b, λ)	0.1483	0.7589	0.0935 (0.3461)	5.9605 (0.8218)	34.5462 (61.141)	0.0615 (0.1021)	
Ga (a, b)	0.1480	0.7572	0.0934 (0.3471)	5.9526 (0.8193)	2.2708 (0.3261)		
APEx (α, λ)	0.1843	0.9396	0.0954 (0.3228)	34031.74 (12329.98)	1.0996 (0.0495)		
TGEx (α, λ, θ)	0.1875	0.9581	0.0966 (0.3078)	6.18734 (1.9283)	1.1019 (0.0947)	-0.6837 (0.2793)	
EEx (α, λ)	0.2267	0.1859	0.1077 (0.1962)	7.7882 (1.4962)	1.0132 (0.0875)		
Ex (λ)	0.1493	0.7643	0.3206 (0.0000)	0.3815 (0.0381)			

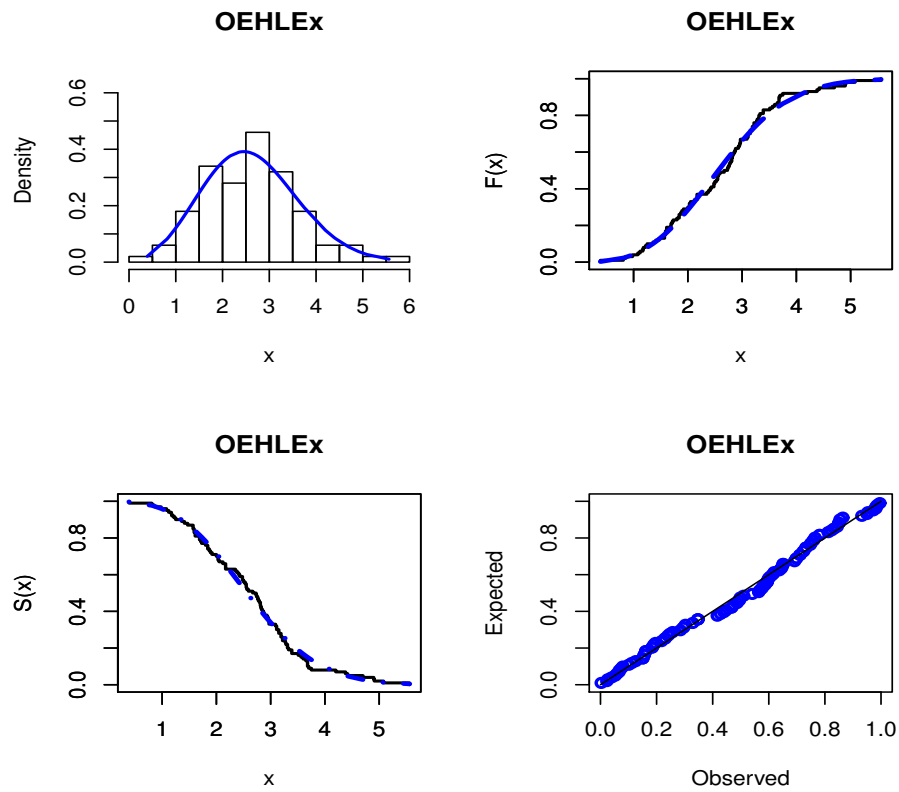


Fig. 4. Fitted PDF, CDF, SF and PP plots of the OEHLEx model for data set II.

6. Conclusions

We propose a new *odd exponentiated half-logistic exponential* (OEHLEx) distribution with two extra shape parameters. The OEHLEx density function can be expressed as a linear mixture of exponential densities. We provide some of its mathematical properties including explicit expressions for the quantile and generating functions, ordinary and incomplete moments, mean residual life and mean inactivity time. The model parameters have been estimated by the maximum likelihood estimation method. We assess the performance of the maximum likelihood estimators via a simulation study. Two applications illustrate that the proposed distribution provides consistently better fits than other competitive models generated using well-known classes.

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