The Persistence of a Discrete Host-parasitoid Model with Allee Effect and Holling III Functional Response

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Abstract—This paper establishes a host-parasitoid model with Allee effect and Holling III functional response, and discusses the persistent problem of this system. Through computer simulation, the growth rate of r is used as the parameter of the bifurcation diagram, and the effect of Allee effect on the dynamic behavior of the system is analyzed.

Keywords—Allee effect; Holling III functional response; bifurcation diagram; persistence

I. INTRODUCTION

A lot of researchers have been interested in the research of Allee effect in hot topics of ecological research[1-3]. A host-parasitoid model with a Allee effect for the host is given by [4] as follows:

$$\begin{cases} H(t+1) = H(t) \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} - \frac{abP(t)}{1+aH(t)} \right] \\ P(t+1) = H(t) \left[1 - \exp \left(-\frac{abP(t)}{1+aH(t)} \right) \right] \end{cases} \quad (1)$$

This paper studies a host-parasitoid model with a Allee effect and Holling III functional response as follows:

$$\begin{cases} H(t+1) = H(t) \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} - \frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right] \\ P(t+1) = H(t) \left[1 - \exp \left(-\frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right) \right] \end{cases} \quad (2)$$

where $H(t)$ is the host population size in generation t , $P(t)$ is the parasitoid population size in generation t , T is the total time for effective search, T_n is the processing time r is the intrinsic growth rate. K is the carrying capacity of the environment and n is the Allee effect for the host. Many species abide by their own ecological genetic rules, and all of them have their own Allee effect. Once the population size goes below the lower bound, the species would die out. Therefore, it is necessary to investigate the interspecific interaction with a lower bound. In this model, these parameters r, b, T, c, m, n and T_n are all positive constants.

II. PERMANENCE

A. The Basic Concept

First, the definition of persistence and the related lemma are given.

Theorem 1[5]. There exist positive constants M_1, M_2 which are independent of the solutions of system, such that for any positive solution $\{N_t, P_t\}$ of system with initial condition, one has:

$$\begin{aligned} M_1 &\leq \liminf_{t \rightarrow \infty} N_t \leq \limsup_{t \rightarrow \infty} N_t \leq M_2, \\ M_1 &\leq \liminf_{t \rightarrow \infty} P_t \leq \limsup_{t \rightarrow \infty} P_t \leq M_2, \quad t = 1, 2, \dots \end{aligned}$$

the system is persistent.

Now we state several lemmas which will be useful in proving our main result.

Lemma 1[5]. Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and $x(k+1) \leq x(k) \exp\{a - bx(k)\}$ for $k \in \mathbb{N}$, where a and b are positive constants, then $\limsup_{k \rightarrow \infty} x(k) \leq \frac{1}{b} \exp(a-1)$.

Lemma 2[5]. Assume that $\{x(k)\}$ satisfies

$$x(k+1) \geq x(k) \exp\{a - bx(k)\}, k \geq N_0,$$

$\limsup_{k \rightarrow \infty} x(k) \leq x^*$ and $x(N_0) > 0$, where a and b are

positive constants, and $N_0 \in \mathbb{N}$. Then

$$\liminf_{k \rightarrow \infty} x(k) \geq \min \left\{ \frac{a}{b} \exp(a - bx^*), \frac{a}{b} \right\}.$$

B. Main Result

Next, we give the proof of the system (2) persistence theorem. Assume that $\{H(t), P(t)\}$ is the solution of the system(2).

Theorem 1. If the system (2) is under the condition of

$$\limsup_{t \rightarrow \infty} H(t) \leq N, \limsup_{t \rightarrow \infty} P(t) \leq N,$$

$$\liminf_{t \rightarrow \infty} H(t) \geq M^*, \liminf_{t \rightarrow \infty} P(t) \geq M^*,$$

and

$$M_1 \leq M^* \leq \liminf_{t \rightarrow \infty} H(t) \leq \limsup_{t \rightarrow \infty} H(t) \leq N \leq M_2,$$

$$M_1 \leq M^* \leq \liminf_{t \rightarrow \infty} P(t) \leq \limsup_{t \rightarrow \infty} P(t) \leq N \leq M_2,$$

the system is persistent.

Proof(1). Prove $\limsup_{t \rightarrow \infty} H(t) \leq N$, $\limsup_{t \rightarrow \infty} P(t) \leq N$, of which $\frac{Ke^{(rm/K+rn/K+r-1)}}{r} = N$.

Proof. For any $t \geq 0$ of $H(t) > 0$, $P(t) > 0$, and $r > 0, n > 0, K > 0, m > 0, k > 0, T > 0, T_n > 0, c > 0$, by the system parameters, the first equation of the system (1) can be derived:

$$\begin{aligned} H(t+1) &= H(t) \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} - \frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right] \\ &\leq H(t) \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} \right] \\ &= H(t) \exp \left[\frac{(r(1-H(t)/K)(H(t)-n))(H(t)+m)}{(H(t)+m)(H(t)+m)} \right] \\ &\leq H(t) \exp \left[\frac{-\frac{r}{K}H(t)(H(t)+m) + \left(\frac{rm}{K} + \frac{rn}{K} + r\right)(H(t)+m) - \left(\frac{rm^2}{K} + \frac{rnm}{K} + rm + rn\right)}{H(t)+m} \right] \\ &\leq H(t) \exp \left[\frac{rm}{K} + \frac{rc}{K} + r - \frac{r}{K}H(t) \right], \end{aligned}$$

By lemma 1:

$$\limsup_{t \rightarrow \infty} H(t) \leq \frac{Ke^{(rm/K+rn/K+r-1)}}{r} = N.$$

Similarly, according to the second formula of system:

$$\begin{aligned} P(t+1) &= H(t) \left[1 - \exp \left(-\frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right) \right] \leq H(t), \\ &\leq H(t-1) \exp \left[\left(\frac{rm}{K} + \frac{rn}{K} + r \right) - \left(\frac{r}{K} \right) H(t-1) \right], \end{aligned}$$

$$\text{Thus } \limsup_{t \rightarrow \infty} P(t) \leq \frac{Ke^{(rm/K+rn/K+r-1)}}{r} = N.$$

Proof (2). It is proved that if $\left(\frac{rm^2}{K} - nm - rm - \frac{rnm}{K} \right) > 0$, there are $\liminf_{t \rightarrow \infty} H(t) \geq M^*$, $\liminf_{t \rightarrow \infty} P(t) \geq M^*$, of which

$$\begin{aligned} (n-m+K) \exp \left[\frac{rn}{K} - \frac{rm}{K} + r - \frac{r}{K}N \right] \exp \left(-\frac{bTN}{2\sqrt{bT_n+c}} \right) &= M^* \\ \sqrt{\frac{(1 + \exp \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]) \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]}{bT}} &= M^*. \end{aligned}$$

Proof. The proof of (1) in $\limsup_{t \rightarrow \infty} H(t) \leq N$, $\limsup_{t \rightarrow \infty} P(t) \leq N$, can be obtained $\forall \varepsilon_1, \exists t_1 > 0$, when $t_1 > t$, has $H(t) \leq N + \varepsilon_1, P(t) \leq N + \varepsilon_1$, the first equation of the system (1) can be derived:

$$\begin{aligned} H(t+1) &= H(t) \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} - \frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right] \\ &= H(t) \exp \left[\frac{rH(t) - \frac{rH^2(t)}{K} + \frac{r}{K}nH(t) - rn}{H(t)+m} \right] \exp \left(-\frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right) \\ &\geq H(t) \exp \left[\frac{rH(t) - \frac{rH^2(t)}{K} + \frac{r}{K}nH(t) - rn}{H(t)+m} \right] \exp \left(-\frac{bTP(t)}{2\sqrt{bT_n+c}} \right) \\ &\geq H(t) \exp \left[r - \frac{r(H(t)+m)}{K} + \frac{r}{K}n \right] \exp \left(-\frac{bT(N+\varepsilon_1)}{2\sqrt{bT_n+c}} \right), \end{aligned}$$

By lemma 2:

$$\liminf_{t \rightarrow \infty} H(t) \geq \frac{\frac{m}{K} - \frac{rm}{K} + r}{\frac{r}{K}} \exp \left(\frac{rm}{K} - \frac{rm}{K} + r - \frac{r}{K}(N+\varepsilon_1) \right) \exp \left(-\frac{bT(N+\varepsilon_1)}{2\sqrt{bT_n+c}} \right)$$

If $\varepsilon_1 \rightarrow 0$, that

$$\liminf_{t \rightarrow \infty} H(t) \geq (n-m+K) \exp \left[\frac{rm}{K} - \frac{rm}{K} + r - \frac{r}{K}N \right] \exp \left(-\frac{bTN}{2\sqrt{bT_n+c}} \right) = M^*$$

the second equation of the system (1) can be derived:

$$P(t+1) = H(t) \left[1 - \exp \left(-\frac{bTH(t)P(t)}{1+cH(t)+bT_nH^2(t)} \right) \right],$$

$$Q = \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} \right]$$

$$M = 1 + CH(t) + bT_n H^2(t)$$

So

$$P(t) = \sqrt{\frac{[M](1+Q) \ln \exp \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} \right]}{bT}}$$

$$= \sqrt{\frac{M(1+Q) \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} \right]}{bT}}$$

$$\geq \sqrt{\frac{(1+Q) \left[\frac{r(1-H(t)/K)(H(t)-n)}{H(t)+m} \right]}{bT}}$$

$$\geq \sqrt{\frac{(1 + \exp \left[r - \frac{r(H(t)+m)}{K} + \frac{r}{K}n \right]) \left[r - \frac{r(H(t)+m)}{K} + \frac{r}{K}n \right]}{bT}}$$

In the case of proof (1), $H(t) \leq N$ can be obtained.

So

$$\geq \sqrt{\frac{(1 + \exp \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]) \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]}{bT}}$$

If

$$\sqrt{\frac{(1 + \exp \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]) \left[r - \frac{r(N+m)}{K} + \frac{r}{K}n \right]}{bT}} = M^*$$

that $P(t) \geq M^*$ then: $\liminf_{t \rightarrow \infty} P(t) \geq M^*$.

By proof (1) (2), we can know that the system is persistent.

III. NUMERICAL SIMULATION

Bifurcation diagram is a frequently used method in Mathematical Research of dynamic systems. Bifurcations will appear in continuous systems or discrete systems. Bifurcation diagrams can observe the small and continuous changes of system parameters (bifurcation parameters), resulting in the sudden change of system essence or topology. In this paper, the intrinsic growth rate of r is used as the parameter of the bifurcation diagram to simulate its influence on the dynamic behavior of the model, and the changes of the population number can be seen visually through the graphics[6].

A. The Effect of Allee Effect on Model Dynamics

Through Matlab simulation, we can see the potential influence of Allee effect on model generated by figure 1 (a) and figure 1 (b).

FIGURE I. MODEL (2) WITH HOLLING III EFFECT, WITHOUT ALLEE EFFECT

Figure 1 is a model bifurcation diagram with no Allee effect under the condition of the Holling III functional response function, figure 1 is a model bifurcation diagram with Allee effect under the condition of the Holling III functional response function. Similarly, the intrinsic growth rate r as a bifurcation diagram, and the number of parasitic species P in the ordinate..

The initial conditions are as follows: $T_1 = 1, T_h = 0.4, K = 5, a = 0.2, b = 0.1$, where Allee effect is not included as shown in figure 1 ($n = 0, m = 0$), where Allee effect is included as shown in figure 2 ($n = 0.5, m = 0.1$). The comparison of the two diagrams can be found: with Allee effect, The maximum number of parasite population decreased from 35 to 22, when $r < 3.48$, whether or not Allee is added, the change of cycle and chaos is almost the same, the left shift of population dynamic changes after the addition of Allee effect, sudden extinction at $r > 3.48$. Therefore, it can be concluded that Allee effect is also a factor affecting the dynamic change of the model. Considering the Holling III functional response function, the population will be destroyed by the Allee effect and the whole system will collapse.

FIGURE II. MODEL (2) WITH HOLLING III EFFECT AND ALLEE EFFECT

IV. CONCLUSION

A discrete host-parasitoid system model with Holling III functional response function and Allee effect was established in this paper. The system was simulated with Matlab software,

and the models of the Holling III functional response and Allee effect were given respectively. We make a comparative analysis.

In this paper, the bifurcation diagrams with Allee effect and no Allee effect are simulated in the discrete host-parasitoid model of Holling III. By comparison, it is found that the population will eventually become extinct after the introduction of the Allee effect. The bifurcation diagram of the model system with the Allee effect and the Holling III functional response at the same time shows that although the introduction of Allee effect eventually leads to the extinction of the population, the influence of the Allee effect on the trend of the whole dynamic change is not very great for the dynamic process of the model change, and it can be seen that after the effect of the Allee effect is strengthened, it can be seen that the effect of the Allee effect is made. The faster the group is extinct is the strong Allee effect.

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