Parameter Estimation of Exponential Distribution

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Abstract—In this paper a new parameter estimation method is given in view of another common situation of exponential distribution (stipulate monitor time, in each monitor time observing the number of expired sample, namely do not observe exact time). The properties of this estimate is also discussed. Simulation is given to illustrate the feasibility of this method.

Keywords—multinomial distribution; exponential distribution; dichotomy

I. INTRODUCTION

Exponential distribution is one of the most widely used distributions. The life of electronic components is usually distributed as exponential distribution. When failure rate function is known, its distribution function is completely determined. The simplest failure rate function is constant function and the distribution it determined is exponential distribution. We denote $E(\lambda)$ as the exponential distribution whose parameter is λ . It is commonly called single-parameter exponential distribution which can distinct with double-parameter exponential distribution. Here we give the simple introduction about two type of censoring data. Type I censoring occurs if an experiment has a set number of subjects or items and stops the experiment at a predetermined time, at which point any subjects remaining are right-censored. Type II censoring occurs if an experiment has a set number of subjects or items and stops the experiment when a predetermined number are observed to have failed.

λ is commonly unknown in exponential distribution and need to be estimated from the observed samples. For completed data, type II censoring data and type I censoring data there existed related results. However, there are also some cases for a batch of experimental samples we can't observe them all the time but only observe them at intervals, that is to say, it is impossible to observe the exact lifetime of the sample and only observe the number of samples failure in each period. This paper aims to give the common formula of failure rate λ 's estimation.

II. MLE(MAXIMUM LIKELIHOOD ESTIMATION) OF λ

Now we sample n products from electronic components to examine their lifetime. We assume the lifetime is distributed as $E(\lambda)$. Let $X_1, X_2, \dots, X_n$ denote the lifetime of these products, respectively which means $X_i \sim E(\lambda)$, $i=1, \dots, n$. Now we stipulate monitor time $t_1, t_2, \dots, t_m (t_1 < t_2 < \dots < t_m)$ and observe the number of expired sample at every time point. The observed sample denoted as $Y_1, Y_2, \dots, Y_m$. The observed value

is $n_1, n_2, \dots, n_m$, n_i is the number of expired sample in $[t_{i-1}, t_i]$. In this case the common method of estimating λ is MLE. Because the lifetime of every electronic component may fall into any time period of m time period, so $Y_1, Y_2, \dots, Y_m$ is distributed as multinomial distribution. Let p_i denote the probability of sample falling into $[t_{i-1}, t_i]$, then the likelihood function of the sample is

$$L(\lambda; n_1, n_2, \dots, n_m) = p(Y_1 = n_1, Y_2 = n_2, \dots, Y_m = n_m) \\ = \frac{(n_1 + n_2 + \dots + n_m)!}{n_1! n_2! \dots n_m!} p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}, \quad (1)$$

where

$$p_i = p(t_{i-1} < x_j < t_i) = F(t_i) - F(t_{i-1}) = e^{-\lambda t_{i-1}} - e^{-\lambda t_i}, \quad j=1, 2, \dots, n.$$

Take the logarithm we can have

$$\ln L(\lambda; n_1, n_2, \dots, n_m) = \ln \frac{(n_1 + n_2 + \dots + n_m)!}{n_1! n_2! \dots n_m!} + \sum_{i=1}^m n_i \ln p_i \\ = \ln \frac{(n_1 + n_2 + \dots + n_m)!}{n_1! n_2! \dots n_m!} + \sum_{i=1}^m n_i \ln(e^{-\lambda t_{i-1}} - e^{-\lambda t_i}). \quad (2)$$

The likelihood equation is

$$\frac{\partial}{\partial \lambda} \ln L(\lambda; n_1, n_2, \dots, n_m) = \sum_{i=1}^m n_i \frac{t_i e^{-\lambda t_i} - t_{i-1} e^{-\lambda t_{i-1}}}{e^{-\lambda t_{i-1}} - e^{-\lambda t_i}} = 0. \quad (3)$$

Solve this equation then we can have MLE of λ .

Many likelihood equations don't have explicit solution. The likelihood equation above belongs to this class. Denote the left side of equation (3) as $f(\lambda)$. It is easy to see that, $\lim_{\lambda \downarrow 0} f(\lambda) > 0, \lim_{\lambda \uparrow 1} f(\lambda) < 0$. Therefore, for given $t_1, t_2, \dots, t_m, n_1, n_2, \dots, n_m$, we can use dichotomy to get the value of λ ($0 < \lambda < 1$).

III. EXAMPLE

In equation (3) we give the general formula that the estimate of λ satisfied in exponential distribution. Next the correctness of this formula is verified by an example. First we generate data from $E(0.01)$. The following is the data we generate from $E(0.01)$ using R software:

[1] 117.7572 30.3796 97.77 69.45 7607 39.98
135.29 13.04 81.48 0.71

[11] 29.3824 86.4223 30.56 84.53 193.6 50.66
38.36 128.10 28.3 31.1

[21] 2.3824 5.9624 59.69 94077 45.54 45.51 50.20
362.79 147.25 54.8.

Assume the observed time are $t_1 = 20, t_2 = 60, t_3 = 100, t_4 = 200$, then we can obtain that the observed value of sample are $n_1 = 4, n_2 = 13, n_3 = 7, n_4 = 5$. The likelihood equation is

$$4 \times \frac{20e^{-20\lambda}}{1 - e^{-20\lambda}} + 13 \times \frac{60e^{-60\lambda} - 20e^{-20\lambda}}{e^{-20\lambda} - e^{-60\lambda}} + 7 \times \frac{100e^{-100\lambda} - 60e^{-60\lambda}}{e^{-60\lambda} - e^{-100\lambda}} + 5 \times \frac{200e^{-200\lambda} - 100e^{-100\lambda}}{e^{-100\lambda} - e^{-200\lambda}} = 0.$$

We can get the value of λ using dichotomy. Use R software to calculate and the codes are as follows:

```
a = 0; b = l; v = 0; jindu = 0.001;
> findrootv<-function(a, b, v, jindu){if ((rootf(a)-
v)*(rootf(b)-v)>0) return("no root")
+ er=abs(rootf(a)-rootf(b)); l=a; u=b;
+ while(er>jindu){m=0.5*(u+l);
+ if((rootf(l)-v)*(rootf(m)-v)>0) l=m else
u=m; er=abs(rootf(l)-rootf(u))
+ root=0.5*(u+l); f=rootf(root);
+ return( c(root, f))
> rootf<-function(x){4*20*exp(-20*x)/(1-exp(-
20*x))+13*(60*exp(-60*x)-20*exp(-20*x))/(exp(-20*x)-exp(-
60*x))
++7*(100*exp(-100*x)-60*exp(-60*x))/(exp(-60*x)-exp(-
100*x))+5*(200*exp(-200*x)-100*exp(-100*x))/(exp(-
100*x)-exp(-200*x))+;
> z<-findrootv(a, b, v, jindu)
> z
[1] 0.0095078461 -0.0001076771.
```

From the solution we can see that the estimate of λ is basically the same as the true value.

IV. LARGE SAMPLE PROPERTIES OF λ

From section 2 we know that MLE of λ satisfy the following equation

$$\sum_{i=1}^m \frac{n_i}{p_i(\lambda)} \frac{\partial p_i(\lambda)}{\partial \lambda} = 0, \lambda \in \Theta \subset R. \quad (4)$$

Let $\vec{P}(\lambda) = (p_1(\lambda), p_2(\lambda), \dots, p_m(\lambda))'$, next we discuss consistency and asymptotic normality.

Theorem 1 Assume (1) Θ is the open subset of R ; (2) for any two points $\lambda_1, \lambda_2 \in \Theta, \lambda_1 \neq \lambda_2, \vec{P}(\lambda_1) \neq \vec{P}(\lambda_2)$; (3)

$\frac{\partial p_i(\lambda)}{\partial \lambda}$ exist and is continuous in $\Theta, i = 1, 2, \dots, m$; (4)

$A(\lambda) = (\frac{\partial \vec{P}(\lambda)}{\partial \lambda}) = (\frac{\partial p_i(\lambda)}{\partial \lambda}), rank(A(\lambda)) = 1, \forall \lambda \in \Theta$.

Then $\hat{\lambda}_n$ is consistent estimation of λ , and

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \rightarrow N(0, I^{-1}(\lambda)). \quad (5)$$

where $\hat{\lambda}_n$ is MLE of λ .

$$\text{Let } M(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{p_1(\lambda)}} \frac{\partial p_1(\lambda)}{\partial \lambda} \\ \dots \\ \frac{1}{\sqrt{p_m(\lambda)}} \frac{\partial p_m(\lambda)}{\partial \lambda} \end{pmatrix},$$

Then $I(\lambda) = M'(\lambda)M(\lambda)$.

Obviously, every term in equation (4) satisfy the conditions of Theorem 1. Therefore, $\hat{\lambda}$ obtained from (3) is the consistent estimation of λ and is asymptotically normal, that is,

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N(0, I^{-1}(\lambda)),$$

where $I(\lambda) = M'(\lambda)M(\lambda)$

$$= t_1^2 e^{-2\lambda t_1} \frac{1}{1 - e^{-\lambda t_1}} + \frac{(t_2 e^{-\lambda t_2} - t_1 e^{-\lambda t_1})^2}{e^{-\lambda t_1} - e^{-\lambda t_2}} + \dots + \frac{(t_m e^{-\lambda t_m} - t_{m-1} e^{-\lambda t_{m-1}})^2}{e^{-\lambda t_{m-1}} - e^{-\lambda t_m}}.$$

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