

# Progress of Research on Weak Hopf Algebra

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**Abstract**—Hopf algebra concept of the early 1940s, the abstract algebraic topology home H. Hopf made on the basis of the work developed in the study of manifolds, since J. W. Milnor and J. C. Moore written entitled “On the structure of post-Hopf algebras” article published in 1965, Hopf algebra began as a branch of algebra gradually been attention and study. In this paper, weak Hopf algebra were discussed in detail, studies were reviewed progress on some aspects, aims to study some open problems and provides useful guidance and reference for weak Hopf algebra.

**Keywords**—Hopf algebra; progress; structure; guidance

## I. INTRODUCTION

G. Bohm and K. Szlachanyi introduce the concept of weak Hopf algebra generalizes the usual Hopf algebra. Simply put, a weak Hopf algebra is a vector space  $k$  domain, Hopf algebra has the same definition criteria, but weakening the Hopf algebra the remainder multiplication units and to plot more than guidelines. There are examples of weak Hopf algebra group algebra, algebraic surfaces, quantum groups and generalized Kac algebra [1,2]. The following chart shows concept of Hopf algebra.

G. Bohm, F. Nill and K. Szlachanyi studied from the perspective of algebra Hopf algebra, the establishment of a general theory of weak Hopf algebra. Many classical Hopf algebra results can be generalized to Hopf algebra. G. Bohm, D. Nikshych, K. Szlachanyi and other weak Huobu Fu algebra to do a more in-depth research. The master's thesis on the basis of original research on existing achievements, get some conclusions [3].

## II. PRELIMINARIES

### A. Definition and Main Conclusions of Weak Hopf Algebra

Definition set  $H = \langle H, \mu, \eta, \Delta, \varepsilon \rangle$  is a combination of algebra and algebra over a field  $k$ , called dual algebra  $H$  is weak, if  $H$  under the condition:

$$(1) \Delta(xy) = \Delta(x)\Delta(y) \quad \forall x, y \in H$$

$$(2) \Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)), \Delta^2(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

$$\text{Among them } \Delta^2 = (\Delta \otimes I) \circ \Delta$$

$$(3) \varepsilon(xyz) = \sum \varepsilon(xy_i) \varepsilon(y_2z)$$

$$\varepsilon(xyz) = \sum \varepsilon(xy_2) \varepsilon(y_1z), \forall x, y, z \in H$$

If the presence of  $S \in \text{End}(H)$ , the following conditions:

$$(4) (i) \sum x_1 S(x_2) = \sum \varepsilon(1_1 x) 1_2,$$

$$(ii) \sum S(x_1) x_2 = \sum 1_1 \varepsilon(x 1_2)$$

$$(iii) \sum S(x_1) x_2 S(x_3) = S(x)$$

Among them  $\Delta(1) = \sum 1_1 \otimes 1_2$ ,  $H$  has called for

extremely weak weak Hopf algebra  $S$ .

Note: Let  $H$  Hopf algebra, the following conditions are equivalent:

(1)  $H$  is a Hopf algebra.

$$(2) \Delta(1) = 1 \otimes 1$$

$$(3) \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \forall x, y \in H$$

$$(4) \varepsilon(1, x) 1_2 = 1_1 \varepsilon(x 1_2) = \varepsilon(x), \forall x \in H$$

### B. Integration of Weak Hopf Algebra

Definition Let  $H$  is a weak Hopf algebra  $r \in H$  is called a left (right) points, if  $\forall h \in H$  the following conditions are met:

$$h \bullet r = \varepsilon_i(h) r (h \bullet r = r \varepsilon_s(h))$$

Let  $r \in H$  be a left integral, called  $r$  is a non-degenerate left integral, if it can be defined by  $H^*$  on a non-degenerate functions.

## III. THE LATEST RESEARCH RESULTS

### A. Domestic Research Results

Weak Hopf algebras have been proposed as a new generalization of ordinary Hopf algebras that appeared in relation to integrable spin chains and classification of subfactors of von Neumann algebras. In contrast to other Hopf algebraic constructions such as the quasi-Hopf algebras weak Hopf algebras are coassociative [4-7].

Yin Yanmin studied the center construction and fundamental theorem of weak hopf algebras. The category DHM is isomorphic to the center of the category HM. The

fundamental theorem for Hopf modules is generalized to Weak Hopf algebras [8].

Braided tensor categories were introduced by A Joyal and R Streets. Algebraic structures within them, especially Hopf algebras were introduced by S Majid. The author Shouchuan Zhang and H Chen constructed the double bicrossproduct  $D = A_\alpha^\phi B_\beta^\phi$  in braided tensor categories and gave the necessary and sufficient conditions for D to be a bialgebra [9-11]. Articles published about domestic research results are shown in Figure 2.

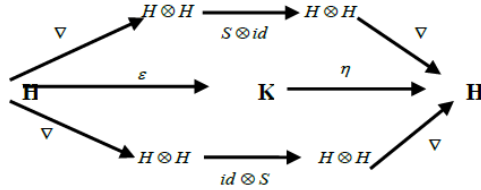


FIGURE I. CONCEPT OF HOPF ALGEBRA

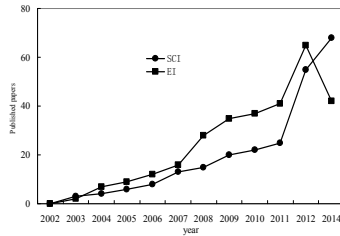


FIGURE II. ARTICLES PUBLISHED ABOUT DOMESTIC RESEARCH RESULTS

### B. Foreign Research Results

Henri Moscovici and Bahram Rangipour associate to each infinite primitive Lie pseudogroup Hopf algebra of ‘transverse symmetries,’ by refining a procedure due to Connes and the first author in the case of the general pseudogroup. J.N. Alonso Álvarez et al. prove that if  $g : B \rightarrow H$  is a morphism of weak Hopf algebras which is split as an algebra-coalgebra morphism, then the subalgebra of coinvariants  $B_H$  of B is a Hopf algebra in the category of Yetter–Drinfeld modules associated to H. Dmitri Nikshych studied the group of group-like elements of a weak Hopf algebra and derive an analogue of Radford’s formula for the fourth power of the antipode S; Gabriella Böhmer et al. give an introduction to the theory of weak Hopf algebras proposed as a coassociative alternative of weak quasi-Hopf algebras. Contents of weak quasi-Hopf algebras is shown in Figure 3.

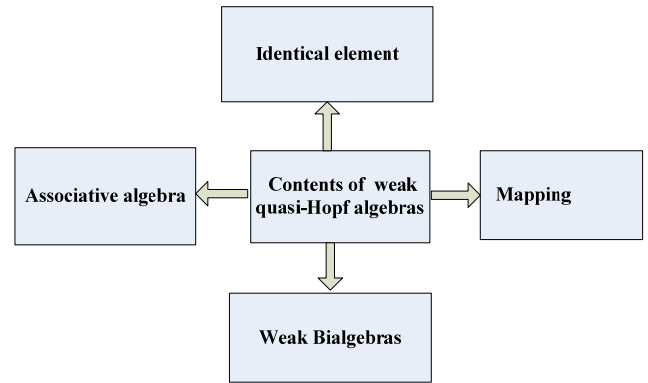


FIGURE III. CONTENTS OF WEAK QUASI-HOPF ALGEBRAS

Lars Kadison and Dmitri Nikshych studied a symmetric Markov extension of  $k$ -algebras  $N \rightarrow M$ , a certain kind of Frobenius extension with conditional expectation that is tracial on the centralizer and dual bases with a separability property. Under this condition, we prove that  $N \rightarrow M$  is the invariantsubalgebra pair of a weak Hopf algebra action by A, i.e., that  $N = M^A$ . The endomorphism algebra  $M_1 = \text{End}_N M$  is shown to be isomorphic to the smash product algebra  $M \# A$ . J.N. Alonso Álvarez et al. study weak Hopf algebras with projection. If  $f : H \rightarrow B$ ,  $g : B \rightarrow H$  are morphisms of weak Hopf algebras such that  $g \circ f = \text{id}_H$ , we prove that it is possible to find an object  $B_H$  in the new category of weak Yetter–Drinfeld modules, that verifies similar conditions to the ones include in the definition of weak Hopf algebra. Finally, they define weak smash bialgebra structures and prove that, under central and cocentral conditions,  $B_H$  and  $H$  determine an example of them. Dmitri Nikshych developed the theory of semisimple weak Hopf algebras and obtains analogues of a number of classical results for ordinary semisimple Hopf algebras. We prove a criterion for semisimplicity and analyze the square of the antipode  $S^2$  of a semisimple weak Hopf algebra A. They explain how the Frobenius–Perron dimensions of irreducible A-modules and eigenvalues of  $S^2$  can be computed using the inclusion matrix associated to A. A trace formula of Larson and Radford is extended to a relation between the categorical and Frobenius–Perron dimensions of A. Finally, an analogue of the Class Equation of Kac and Zhu is established and properties of A-module algebras and their dimensions are studied.

J. N. Alonso Álvarez and R. González Rodríguez proved that if  $g : B \rightarrow H$  is a morphism of weak Hopf algebras which is split as a coalgebra morphism, then there exists a subalgebra  $B_{\text{hof}}$  of B, morphisms

$\varphi_{BH} : H \otimes B_H \rightarrow B_H$ ,  $\sigma_{BH} : H \otimes B_H \rightarrow B_H$  and an isomorphism of algebras and right H-comodules  $b_B : B \rightarrow B_H \times H$ , being  $B_H \times H$  a subobject of  $B_H \otimes H$  with its algebra structure defined by a crossed product. Also, they obtain the dual results and as a consequence we prove the Radford’s theorem for weak Hopf algebras. J.N. Alonso Álvarez et al. proved that if  $g : B \rightarrow H$  is a morphism of weak Hopf algebras which is split as an algebra-coalgebra morphism, then the subalgebra of coinvariants  $B_H$  of B is a

Hopf algebra in the category of Yetter–Drinfeld modules associated to  $H$ .

### C. Development Trend of Weak Hopf Algebras

On one hand, the properties of some important structures over weak Hopf algebras will be continued to discuss. For example, the relation between the global dimension of weak smash product algebra  $A \# H$  and the global dimension of the algebra  $A$  will be given; And the notion of weak measuring of weak entwining structure and verify Frobenius properties and Maschke type theorem of weak entwined modules will be introduced. A statistical data result for Hopf algebras is shown in Figure.4.

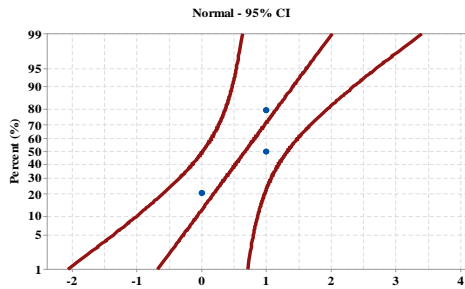


FIGURE IV. A STATISTICAL DATA RESULT FOR HOPF ALGEBRAS

On the other hand, motivated by the introduction of weak Hopf algebra and Hopf group-coalgebra, especially by the fact, that a Hopf group-coalgebra is essentially a Hopf algebra in the symmetric monoidal category, these two different generalizations to introduce the notion of weak Hopf group-coalgebra will be combined. Similar to the work of Hopf group-coalgebra, some important structure in the new content of weak Hopf group-coalgebra will also be generalized.

## IV. THE ISSUES OF WEAK HOPF ALGEBRA

### A. Groups Mold of Alternative Doi-Hopf- weak Hopf Algebra

Because Hopf algebra who played a very important role in quantum groups and related research in mathematical physics, along with the deepening of the research, a number of promotional Hopf algebra is proposed. The most important is the promotion of a weak Hopf algebra (WHA) in introduced its research originated in the expansion of its theory and algebra, contact low-dimensional quantum field theory and its application in the study of Hopf algebra dynamical twists. Roughly speaking, weak Hopf Hopf algebra and algebra have the same structure, in addition to more than the multiplication of keeping units and I keep multiplicative units are weak conditions instead. Therefore, weak Hopf algebra structure than the Hopf algebra complex [12].

Jia Ling first introduced weak Hopf weak mold Alternative Doi-Hopf algebra, then constructed from weak Alternative Doi-Hopf module category to the module category accompanied forget functor functor [13]. Chen Quanguo and Tang Jiangang introduced weak quantum Yetter-Drinfeld modules for weak Hopf group T-coalgebras. Then, it is proved that weak quantum Yetter-Drinfeld modules are special case

of weak Doi-Hopf group-modules. Also, a pair of adjoint functors between the category of weak quantum Yetter-Drinfeld modules and the category of the right B-module is constructed, where B is subalgebra of coinvariance for a weak Hopf group-bicomodule algebra. Finally, the quantum integrals relative to weak quantum Yetter-Drinfeld modules are discussed.

### B. Global Dimension of Weak Smash Product

The global dimension for skew group algebras, or more generally, for smash products and crossed products was discussed by several authors. For example, YANG S L proved that  $gl.dim(A \# H) \leq gl.dim(A)$  if  $H$  is semisimple, where  $gl.dim(A)$  denotes the left global dimension of the algebra  $A$ ; WANG Z X and ZHAO H proved that  $w.dim(A \# H) = w.dim(A)$  under some conditions, where  $w.dim(A)$  denotes the weak global dimension of the algebra  $A$ . YANG Bi-cheng introduced the conception of twisted smash products, which is a generalization of usual smash products.

Zhang Peng studied the global dimension of twisted smash products. The main result is given by the following: If  $H$  is a finite-dimensional cocommutative Hopf algebra such that  $H$  and  $H^*$  are semisimple, then

$$gl.dim(A * H) = gl.dim(A),$$

$$w.dim(A * H) = w.dim(A),$$

where  $A * H$  is the twisted smash product.

## V. CONCLUSION

In this survey it is given a brief account of research achievement on weak Hopf algebra in recent years. Some open problems are reported at the end. Although the date of weak Hopf algebra has been studied in many ways, and references listed at the end of the text is only a small part of the research in these areas. But from the perspective of the theory can be said that some research weak Hopf algebra theorem should be in the future research in one direction; the other latest research trends are: the development of computer hardware and adapt the design is suitable for construction of modern computers, high-performance algebraic methods so as to reflect the level of development in this area of research theory, adapted. Modern applications in real-time computing are needed.

At the same time these theories mathematical development in recent generations, due to the widespread use of electronic technology and computer, and some of the results and methods of the past generations of mathematics can be applied directly to the engineering techniques, such as algebraic coding science, language of algebra, algebra automata new applications in the field of algebra theory have been produced and developed. At the same time it is an important part of discrete mathematics, and has a significant impact on projecting a combination of mathematics and flourish. These new applications promote the

formation of the modern applications of algebra, development and improvement.

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