Optimal Hohmann-Type Impulsive Ellipse-to-Ellipse Coplanar Rendezvous

Xiwen Tian, Yingmin Jia

The Seventh Research Division and the Center for Information and Control, School of Automation Science and Electrical Engineering, Beihang University (BUAA), 37 Xueyuan Road, Haidian District, Beijing, 100191, China

E-mail: tianxiwen123@163.com, ymjia@buaa.edu.cn

www.buaa.edu.cn

Abstract

This paper devotes to the problem of ellipse-to-ellipse coplanar rendezvous, where the solution and distribution of Hohmann-type optimal impulsive rendezvous are investigated. The analytical relation between the initial states and rendezvous time are derived for Hohmann-type, and the optimal impulse amplitudes are given thereupon. The distribution boundary of Hohmann-type model is obtained according to the Hohmann transfer and Hohmann with coasts. Simulations are demonstrated to analyze the influences of the solution and distribution.

Keywords: Optimal impulsive rendezvous, Hohmann-type rendezvous, ellipse-to-ellipse, optimal distribution.

1. Introduction

Optimal impulsive rendezvous is aimed at obtaining minimum-fuel guidance strategy for spacecraft rendezvous, which has attracted considerable attention. Despite that Lawden’s necessary conditions for optimal impulsive trajectories and Lion’s improving methods for non-optimal trajectories have provided some guidelines to solve the optimal problem where the initial states and rendezvous time are specified, the distributions of optimal models cannot be obtained clearly in these way. So far, only Prussing’s theory of optimal impulsive rendezvous on close circular orbits is complete in its theoretical system, which derives the solutions and distributions of optimal impulsive models by solving the primer vector equations and boundary value problem. A reference frame in mean velocity orbit was built by Frank, and showed better performance in describing the impulse locations and magnitudes than the mean radius orbit in Prussing’s results. Xie focused on the selection of reference frame for optimal impulsive rendezvous, and investigated the effect on the classification, distribution and guidance precision. For the case of elliptic orbit rendezvous, Wang used the state transition matrix given by Yamanaka to calculate the optimal solution of four-impulse model, but the analytical solution and the distribution are difficult to be achieved. Chen studied the ellipse-to-circle coplanar rendezvous based on his results on the dynamical equations for elliptic orbit rendezvous in low eccentricity, and provided the solutions and distributions of all types optimal models. Motivated by which, our previous work considered the ellipse-to-ellipse coplanar rendezvous and obtained the analytical solution and distribution of four-impulse model. In this paper, we will further investigate the Hohmann-type model for optimal impulsive ellipse-to-ellipse coplanar rendezvous.

2. Dynamics Description

Copyright © 2018, the Authors. Published by Atlantis Press.
This is an open access article under the CC BY-NC license (http://creativecommons.org/licenses/by-nc/4.0/).
The relative motion between two spacecrafts in elliptic orbits was derived in our previous work\textsuperscript{10}, which is still used in this paper and given as follow:

\begin{equation}
\begin{bmatrix}
\delta r' = 3 \delta r + 2 \delta \theta \\
\delta \theta = -2 \delta \delta
\end{bmatrix}
\end{equation}

The initial and terminal states of system (1) are

\begin{equation}
x_0 = \begin{bmatrix} x_{01}, x_{02}, x_{03}, x_{04} \end{bmatrix}^T
\end{equation}

\begin{equation}
x_\tau = \begin{bmatrix} 0, 0, 0, 0 \end{bmatrix}^T
\end{equation}

where

\begin{equation}
x_{01} = k \left( 1 + e \cos f_1 \right)^{\alpha_1} - k \left( 1 + e \cos f_1 \right)^{\alpha_1}
x_{02} = \beta
\end{equation}

\begin{equation}
x_{03} = k \frac{1}{2} e \cos f_1 - k \frac{1}{2} e \sin f_1
\end{equation}

\begin{equation}
x_{04} = k \frac{3}{2} \left( 1 + e \cos f_1 \right)^{\alpha_2} - k \frac{3}{2} \left( 1 + e \cos f_1 \right)^{\alpha_2}
\end{equation}

$\beta$ is the difference of phase angle between two spacecrafts; $e_1, e_2$ and $f_1, f_2$ are their eccentricities and true anomalies, respectively.

The states at phase angle $\tau$ was also deduced\textsuperscript{10}:

\begin{equation}
x(\tau) = \begin{bmatrix}
2d_1 - d_2 \cos(\tau + \phi) \\
\beta - 3d_1 \tau + 2d_1 \sin(\tau + \phi) - 2d_1 \sin \phi \\
d_1 \sin(\tau + \phi) \\
-3d_1 + 2d_1 \cos(\tau + \phi)
\end{bmatrix}
\end{equation}

where

\begin{equation}
d_i = x_{03} + x_{04} \quad d_4 = 2x_{01} + x_{04}
\end{equation}

\begin{equation}
d_3 = \sqrt{d_1^2 + d_2^2}
\end{equation}

\begin{equation}
d_2 = 2x_{01} + x_{04}, \phi = \arcsin \left( \frac{d_1}{d_3} \right)
\end{equation}

3. Optimal Hohmann-Type Rendezvous

The solution to primer vector equations corresponding to system (1) can be given in the following form:

\begin{equation}
\begin{bmatrix}
\lambda_1 = A \cos \tau + B \sin \tau + 2C \\
\lambda_2 = 2B \cos \tau - 2A \sin \tau - 3C \tau + D
\end{bmatrix}
\end{equation}

Hohmann-type model is a special case of optimal two-impulse rendezvous, where the coefficients of (6) are

\begin{equation}
A = B = C = 0, D = \pm 1
\end{equation}

then $\lambda_1 = 0, |\lambda_2| = 1$. It can be verified that the necessary conditions of optimal impulsive rendezvous are satisfied for any phase angle $\tau$.

3.1. Solution of Hohmann Transfer

The impulse direction can be obtained from the solution (6), while the impulse time and magnitudes needed be calculated according to the following boundary value problem:

\begin{equation}
\begin{bmatrix}
2(1-C_\tau) \\
4S_\tau - 3r_f \\
2S_\tau \\
-3 + 4C_\tau
\end{bmatrix}
\end{equation}

where $S_\tau = \sin \tau, C_\tau = \cos \tau, \Delta V_1$ and $\Delta V_2$ are the magnitudes of two impulse, and $r_f$ is the rendezvous time. From (4) and (8), we have

\begin{equation}
\Delta V_1 = -\frac{d_1 \sin (r_f + \phi)}{2S_\tau}
\end{equation}

\begin{equation}
\Delta V_2 = \frac{2d_1 + d_1 \cos (r_f + \phi)}{2(1-C_\tau)}
\end{equation}

then

\begin{equation}
d_3 \left( \sin (r_f + \phi) - \sin \phi \right) = 2d_4 \sin r_f
\end{equation}

Substituting (5) into (10), it can be obtained that

\begin{equation}
\sqrt{x_{01}^2 + d_1^2} \sin (r_f + \theta) = -d_i
\end{equation}

where

\begin{equation}
\cos \theta = \frac{x_{01}}{\sqrt{x_{01}^2 + d_1^2}}, \sin \theta = \frac{-d_i}{\sqrt{x_{01}^2 + d_1^2}}
\end{equation}

On the other hand, from the second row of (8), it has

\begin{equation}
\Delta V_1 = -\frac{\beta + 3d_1 \tau - 2d_1 \sin (r_f + \phi) + 2d_1 \sin \phi}{(4S_\tau - 3r_f)}
\end{equation}

Combining (5), (9) and (13), it has

\begin{equation}
\frac{3r_f}{2} \left( x_{01}C_\tau + d_1S_\tau \right) + 2d_1 = f_\tau - f_i
\end{equation}

The appropriate initial states and rendezvous time which satisfy the necessary conditions of Hohmann transfer can be obtained by solving (11) and (14) together, and then, the second impulse $\Delta V_2$ can be obtained as

\begin{equation}
\Delta V_2 = -d_1 \frac{3r_f \sin (r_f + \phi)}{2S_\tau}
\end{equation}

3.2. Distribution of Hohmann-type model

The distribution of optimal Hohmann-type rendezvous is to illustrate the existence of feasible solution. To investigate the distribution, rendezvous time is chosen as the X-coordinate and the special phase angle defined below as Y-coordinate:
\[ \delta \theta = -\beta + 1.5d_3 \tau_F - d_3 \sin (\tau_F + \phi) + d_3 \sin \phi \quad (16) \]

Let \( \tau_{ph} \) be rendezvous time solved by (11) and (14), and \( \delta \theta_{ph} \) is the corresponding special phase angle. If \( \tau_F = \tau_{ph} \) and \( \delta \theta_F = \delta \theta_{ph} \), then it is just the Hohmann transfer. The two impulses are implemented at \( \tau_1 = 0 \) and \( \tau_2 = \tau_F \). However, when the real rendezvous time is longer than \( \tau_{ph} \), the coasts are needed to save the fuel.

If \( \tau_F > \tau_{ph} \) and \( \delta \theta_F = \delta \theta_{ph} \), it is a Hohmann model with terminal coast. The two impulses are implemented at \( \tau_1 = 0 \) and \( \tau_2 = \tau_F \), and the residual time \( \tau_F - \tau_{ph} \) is for terminal coast. The special phase angle \( \delta \theta_F \) and rendezvous time \( \tau_F \) should satisfy the following relation

\[ \delta \theta_F = \delta \theta_{ph} + 1.5d_4 (\tau_F - \tau_{ph}) \]
\[ + d_4 \sin (\tau_F + \phi) - d_3 \sin (\tau_F + \phi) \quad (17) \]

If \( \tau_F > \tau_{ph} \) and the special phase angle \( \delta \theta_F \) satisfies

\[ \delta \theta_F = \delta \theta_{ph} - 1.5d_4 (\tau_F - \tau_{ph}) \]
\[ + d_4 \sin (\tau_F - \tau_{ph} + \phi) - d_3 \sin (\tau_F + \phi) \quad (18) \]

then, after the initial coast for time \( \tau_F - \tau_{ph} \), the special phase angle will become exactly \( \delta \theta_{ph} \). This case is a Hohmann model with initial coast, and the impulses are implemented at \( \tau_1 = \tau_F - \tau_{ph} \) and \( \tau_2 = \tau_F \).

Let

\[ \delta \theta_{ph1} = \delta \theta_{ph} + 1.5d_4 (\tau_F - \tau_{ph}) \]
\[ + d_4 \sin (\tau_F + \phi) - d_3 \sin (\tau_F + \phi) \quad (19) \]

\[ \delta \theta_{ph2} = \delta \theta_{ph} - 1.5d_4 (\tau_F - \tau_{ph}) \]
\[ + d_4 \sin (\tau_F - \tau_{ph} + \phi) - d_3 \sin (\tau_F + \phi) \]

If \( \tau_F > \tau_{ph} \) and \( \delta \theta_{ph1} < \delta \theta_F < \delta \theta_{ph2} \), then there exists a Hohmann model with both initial coast and terminal coast. As shown in Fig.1, this case is illustrated in the middle of the curves expressed by (17) and (18), that is the shadow part. Denote \( (\tau_{ph}, \delta \theta_{ph}) \) as the intersection point of the curves determined by (16) and (18), then the two impulses are implemented at \( \tau_1 = \tau_{ph0} - \tau_{ph} \) and \( \tau_2 = \tau_{ph0} \). The initial and terminal coast last for time \( \tau_{ph0} - \tau_{ph} \) and \( \tau_F - \tau_{ph0} \), respectively.

From the above, the optimal Hohmann-type impulsive rendezvous has four models, all of whose impulse magnitudes are determined by (9) and (15), and impulse direction is along the tangential direction.

4. Simulations

In this section, simulation examples are presented to show the guidance performance and distribution of Hohmann-type impulsive rendezvous.

4.1. Hohmann ellipse-to-ellipse rendezvous

It is assumed the semi-major axis and eccentricities of the target orbit and chaser orbit are initially \( a_t = 6730 \) (km), \( a_c = 6750 \) (km), \( e_t = 0.0005 \) and \( e_c = 0.0004 \), respectively. Let \( \tau_F \) and \( \beta \) (rad) be the appropriate rendezvous time and initial difference of phase angle, respectively, which satisfy (11) and (14). And denote \( R(m) \) as the optimal radius of reference frame, \( R(m) \) as the initial relative distance, \( \Delta a(m) \), \( \Delta e \) and \( \Delta \beta \) (rad) as the guidance errors.

\[ \Delta a = 3.62 \quad 3.44 \quad 2.80 \quad 2.68 \]
\[ \Delta e = 7.12e-03 \quad 9.68e-03 \quad 6.04e-03 \quad 3.46e-03 \]
\[ \Delta \beta = 3.38e-05 \quad 4.51e-03 \quad 4.22e-03 \quad 8.92e-06 \]
\[ \Delta R = 1.29e+02 \quad 5.14e+02 \quad 5.71e+02 \quad 5.08e+01 \]

Simulation results of Hohmann transfer for ellipse-to-ellipse rendezvous are demonstrated in Table 1, which shows that: (1) with different true anomalies, even if the other initial states are the same, the
rendezvous time and initial difference of phase angle which satisfy (11) and (14) varies much; (2) the optimal radius of reference frame also changes with the true anomaly; (3) the guidance precision is high when the chaser initially stays around the perigee.

4.2. Distribution of Hohmann-type model

To investigate the distribution of Hohmann-type ellipse-to-ellipse rendezvous, we take rendezvous time $\tau_F$ as the X-coordinate and $\delta \theta_F/d_4$ as Y-coordinate. Fig. 2 shows the distribution of Hohmann-type model with different true anomalies and eccentricities.

4.2.1. Distribution of Hohmann-type model with different true anomalies and eccentricities

(a) $[e_t, e_c] = [0.0004, 0.0005], f_c = 30^\circ$

(b) $[e_t, e_c] = [0.004, 0.005], f_c = 30^\circ$

(c) $[e_t, e_c] = [0.051, 0.05], f_c = 30^\circ$

(d) $[e_t, e_c] = [0.004, 0.005], f_c = 150^\circ$

(e) $[e_t, e_c] = [0.004, 0.005], f_c = 210^\circ$

(f) $[e_t, e_c] = [0.004, 0.005], f_c = 330^\circ$

Fig. 2 Distributions with different true anomalies and eccentricities

5. Conclusion

This paper extends our previous work to the Hohmann-type optimal impulsive rendezvous. By defining the special phase angle, we derived the analytical solution for Hohmann transfer, and obtained that the optimal Hohmann-type impulsive rendezvous has four models, i.e. Hohmann transfer, Hohmann with initial coast, Hohmann with terminal coast and Hohmann with both coasts. In further research, we will integrate all optimal models in one map, including four-
impulse, three-impulse, three-impulse with coasts, two-impulse, two-impulse with coasts, and Hohmann-type.

Acknowledgements
This work was supported by NSFC (61327807, 61521091, 61520106010, 61134005) and the National Basic Research Program of China (973 Program: 2012CB821200, 2012CB821201)

References