Approximation properties of some Lupas-Durrmeyer type operators

Bo-yong Lian [1, a *], Qing-bo Cai[2,b]

[1] Department of Mathematics, Yang-En University, China

[2] School of Mathematics and Computer Science, Quanzhou Normal University, China

[a]lianboyong@163.com, [b]qbcai@126.com

Keywords: Lupas-Durrmeyer type operators; Rate of convergence; Bounded variation functions

Abstract. By using Bojanic-Cheng's method and analysis techniques, the authors study the rate of convergence of Lupas-Durrmeyer type operators for some absolutely continuous functions having a derivative equivalent to a bounded variation.

1 Introduction

In [1],Aral introduced some Lupas-Durrmeyer type operators

$$L_n^{(1/n)}(f,x)=(n+1)\sum_{k=0}^{n}p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)f(t)dt, \qquad (1)$$

where $f\in C[0,1]$, $p_{n,k}^{(1/n)}(x)=\dfrac{2(n!)}{(2n!)}\binom{n}{k}(nx)_k(n-nx)_{n-k}$, $(n)_k=n(n+1)\cdots(n+k-1)$,

$p_{n,k}(t)=\binom{n}{k}t^k(1-t)^{n-k}$, and $p_{n,k}^{(1/n)}(x)$ come from the density function of Polya distribution

$$p_{n,k}^{(\alpha)}(x)=\binom{n}{k}\frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)},x\in[0,1].$$

Recently, Gupta [2] introduced another Lupas-Durrmeyer type operators

$$D_n^{(1/n)}(f,x)=n\sum_{k=1}^{n}p_{n,k}^{(1/n)}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt+p_{n,0}^{(1/n)}(x)f(0), \qquad (2)$$

and investigated the local and global approximaiton properties. Futhermore, the authors also considered the voronvskaya type asymptotic formula. Later, several researchers have made significant contributions in this direction. We refer the reders to some of the related papers [3-6].

The rate of approximaiton for functions with derivatives of bounded variation is an interesting topic. This is mainly originated from Bojanic-Cheng [7], then many scholars have done a lot of research in this field [8-9]. Since the introduction of the operators based on Polya distribution, the work related to this [3-6] has not stopped.

Inspired by this, this article studies the approximation of operator $D_n^{(1/n)}(f,x)$ for some absolutely continuous functions DBV, which having a derivative equivalent to a functions of

bounded function BV.

We get some definition as follows.

Definition 1

$$DBV[0,1]=\left\{f \,\middle|\, f(x)=f(0)+\int_0^x h(t)dt\right\},$$

where $x\in[0,1], h\in BV[0,1]$, i.e., h is a function of bounded variation on $[0,1]$.

Definition 2

$$K_n(x,t)=n\sum_{k=1}^{n} p_{n,k}^{(1/n)}(x)p_{n-1,k-1}(t)+\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

By the Lebesgue-Stieltjes integral representations, we have

$$D_n^{(1/n)}(f,x)=\int_0^1 f(t)K(x,t)dt. \tag{3}$$

2 Some lemmas

We start this section with the following uesful lemmas, which will be used in the sequel.

Lemma 1(see [2]) For $e_i=t^i, i=0,1,2$, we have

$$D_n^{(1/n)}(e_0,x)=1, D_n^{(1/n)}(e_1,x)=\frac{nx}{n+1},$$
$$D_n^{(1/n)}(e_2,x)=\frac{n^2(n-1)x^2+n(3n+1)x}{(n+1)^2(n+2)}.$$

Remark 1 By simple applications of Lemma 1, we get

$$D_n^{(1/n)}(t-x,x)=\frac{-x}{n+1},$$
$$D_n^{(1/n)}((t-x)^2,x)=\frac{(n+2-3n^2)x^2+n(3n+1)x}{(n+1)^2(n+2)}.$$

Remark 2 When n sufficient large, we have

$$D_n^{(1/n)}((t-x)^2,x)\le\frac{3\delta_n^2(x)}{n+1}. \tag{4}$$

where $\delta_n^2(x)=x(1-x)+\frac{1}{n+1}$.

Lemma 2 When n sufficient large, we have

$$D_n^{(1/n)}(|t-x|,x)\le\sqrt{\frac{3}{n+1}}\delta_n(x). \tag{5}$$

Proof. By Cauchy-Schwarz inequality, we have

$$D_n^{(1/n)}(|t-x|,x)\le\sqrt{D_n^{(1/n)}((t-x)^2,x)}\cdot\sqrt{D_n^{(1/n)}(1,x)}\le\sqrt{\frac{3}{n+1}}\delta_n(x).$$

The last inequality is obtained by Lemma 1 and remark 2.

Lemma 3 (i) For $0 \le y < x < 1$, when n sufficient large, there holds

$$R_n(x,y) = \int_0^y K_n(x,t)dt \le \frac{3\delta_n^2(x)}{(n+1)(x-y)^2}.$$

(ii) For $0 < x < z \le 1$, when n sufficient large, there holds

$$1 - R_n(x,z) = \int_z^1 K_n(x,t)dt \le \frac{3\delta_n^2(x)}{(n+1)(z-x)^2}.$$

Proof. (i) By (3) and (4), we get

$$R_n(x,y) = \int_0^y K_n(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n(x,t)dt \le \frac{1}{(x-y)^2}\int_0^1 (t-x)^2 K_n(x,t)dt$$

$$= \frac{1}{(x-y)^2} D_n^{(1/n)}((t-x)^2, x) \le \frac{3\delta_n^2(x)}{(n+1)(x-y)^2}.$$

(ii) Using a similar method, we get (ii) easily.

3 Conclusion

Theorem Let $f \in DBV[0,1]$. If $h(x+), h(x-)$ exist at a fixed point $x \in (0,1)$, when n sufficient large, then we have

$$\left| D_n^{(1/n)}(f,x) - f(x) + \frac{x[h(x+)+h(x-)]}{2(n+1)} \right| \le |h(x+)-h(x-)| \sqrt{\frac{3}{4(n+1)}}\delta_n(x)$$

$$+ \frac{6\delta_n^2(x)}{(n+1)x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (\varphi_x) + \frac{1}{\sqrt{n}} \sum_{k=1}^{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} (\varphi_x).$$

Where

$$\varphi_x(t) = \begin{cases} h(t) - h(x+), & x < t \le 1; \\ 0, & t = x; \\ h(t) - h(x-), & 0 \le t < x. \end{cases}$$

Proof. Let f satisfy the conditions of Theorem, by using Bojanic-Cheng's method [7], we have

$$f(t) - f(x) = \int_x^t h(u), \tag{6}$$

and $h(u)$ can be expressed as

$$h(u) = \frac{h(x+)+h(x-)}{2} + \varphi_x(u) + \frac{h(x+)-h(x-)}{2} sign(u-x) + \delta_x(u)\left[h(x) - \frac{h(x+)+h(x-)}{2}\right], \tag{7}$$

where

$$\delta_x(u)=\begin{cases}1 & , \quad u=x\\ 0 & , \quad u\neq x\end{cases}, \quad sign(x)=\begin{cases}1, & x>0;\\ 0, & x=0;\\ -1, & x<0.\end{cases}$$

From (6) and (7), and noting $\int_x^t sign(u-x)du=|t-x|$, $\int_x^t \delta_x(u)du=0$, we find that

$$D_n^{(1/n)}(f,x)-f(x)=D_n^{(1/n)}(f(t)-f(x),x)=D_n^{(1/n)}(\int_x^t h(u)du,x)$$

$$=\frac{h(x+)+h(x-)}{2}D_n^{(1/n)}(t-x,x)+\frac{h(x+)-h(x-)}{2}D_n^{(1/n)}(|t-x|,x)+D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x).$$

By Remark 1, Remark 2 and Lemma 2, we have

$$\left|D_n^{(1/n)}(f,x)-f(x)+\frac{x[h(x+)+h(x-)]}{2(n+1)}\right|\leq\frac{|h(x+)-h(x-)|}{2}D_n^{(1/n)}(|t-x|,x)+\left|D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)\right|$$

$$\leq|h(x+)-h(x-)|\sqrt{\frac{3}{4(n+1)}}\delta_n(x)+\left|D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)\right|. \quad (8)$$

To complete the proof, we must estimate the term $D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)$.

From (3), the term $D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)$ can be stated as

$$D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)=\int_0^1(\int_x^t \varphi_x(u)du)K_n(x,t)dt=\int_0^1(\int_x^t \varphi_x(u)du)d_tR_n(x,t)$$

$$=\int_0^x(\int_x^t \varphi_x(u)du)d_tR_n(x,t)+\int_x^1(\int_x^t \varphi_x(u)du)d_tR_n(x,t).$$

Let $\Delta_{1n}(f,x)=\int_0^x(\int_x^t \varphi_x(u)du)d_tR_n(x,t), \Delta_{2n}(f,x)=\int_x^1(\int_x^t \varphi_x(u)du)d_tR_n(x,t)$, then we have

$$D_n^{(1/n)}(\int_x^t \varphi_x(u)du,x)=\Delta_{1n}(f,x)+\Delta_{2n}(f,x). \quad (9)$$

Fistly, we estimate $\Delta_{1n}(f,x)$. Using partial integration and noticing $R_n(x,0)=0, \int_x^x \varphi_x(u)du=0$, we get

$$\Delta_1(f,x)=R_n(x,t)\int_x^t \varphi_x(u)du\Big|_0^x-\int_0^x R_n(x,t)\varphi_x(t)dt=-\int_0^x R_n(x,t)\varphi_x(t)dt$$

$$=-\left(\int_0^{x-\frac{x}{\sqrt{n}}}+\int_{x-\frac{x}{\sqrt{n}}}^x\right)R_n(x,t)\varphi_x(t)dt.$$

Thus, it follows that

$$|\Delta_{1n}(f,x)|\leq\int_0^{x-\frac{x}{\sqrt{n}}}R_n(x,t)\overset{x}{\underset{t}{V}}(\varphi_x)dt+\int_{x-\frac{x}{\sqrt{n}}}^x R_n(x,t)\overset{x}{\underset{t}{V}}(\varphi_x)dt.$$

From Lemma 3(i) and $0\leq R_n(x,t)\leq 1$, we have

$$|\Delta_{1n}(f,x)|\leq\frac{3\delta_n^2(x)}{n+1}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\overset{x}{\underset{t}{V}}(\varphi_x)}{(x-t)^2}dt+\frac{x}{\sqrt{n}}\overset{x}{\underset{x-\frac{x}{\sqrt{n}}}{V}}(\varphi_x). \quad (10)$$

Putting $t = x - \frac{x}{u}$ for the integral of (10), we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\overset{x}{\underset{t}{V}}(\varphi_x)}{(x-t)^2} dt = \frac{1}{x}\int_1^{\sqrt{n}} \overset{x}{\underset{x-\frac{x}{u}}{V}}(\varphi_x) du \le \frac{2}{x}\sum_{k=1}^{[\sqrt{n}]} \overset{x}{\underset{x-\frac{x}{k}}{V}}(\varphi_x). \tag{11}$$

From (10) and (11), it follows that

$$\left|\Delta_{1n}(f,x)\right| \le \frac{6\delta_n^2(x)}{(n+1)x}\sum_{k=1}^{[\sqrt{n}]} \overset{x}{\underset{x-\frac{x}{k}}{V}}(\varphi_x) + \frac{x}{\sqrt{n}} \overset{x}{\underset{x-\frac{x}{\sqrt{n}}}{V}}(\varphi_x). \tag{12}$$

Using the same method, we get

$$\left|\Delta_{2n}(f,x)\right| \le \frac{6\delta_n^2(x)}{(n+1)(1-x)}\sum_{k=1}^{[\sqrt{n}]} \overset{x+\frac{1-x}{k}}{\underset{x}{V}}(\varphi_x) + \frac{1-x}{\sqrt{n}} \overset{x+\frac{1-x}{\sqrt{n}}}{\underset{x}{V}}(\varphi_x). \tag{13}$$

Theorem now follows from (8), (9), (12) and (13). This completes the proof.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11601266), the Natural Science Foundation of Fujian Province of China (Grant No.2016J05017), the Program for New Century Excellent Talents in Fujian Province University and the Program for Outstanding Youth Scientific Research Talents in Fujian Province University.

References

[1] A. Aral and V. Gupta, Direct estimates for Lupas-Durrmeyer operators, Filomat, 2016, 30(1): 191-199.
[2] V. Gupta and D. Soybas, Convergence of integral operators based on different distributions, Filomat, 2016, 30(8): 2277-2287.
[3] T. Neer, A M. Acu and P N. Agrawal, Bézier variant of genuine durrmeyer type operators based on polya distribution, Carpathian J.Math., 2016, 33(1): 73-86.
[4] T. Neer and P N. Agrawal, A genuine family of Bernstein-Durrmeyer type operators based on polya basis functions, Filomat, 2017,31(9):2611-2623.
[5] P N. Agrawal, N. Ispir and A. Kajla, Approximation properties of Lupas-Kantorovich operators based on polya distribution, Rendiconti del Circolo Matematico di Palermo Series 2, 2016, 65(2):185-208.
[6] V. Gupta, A M. Acu and D F. Sofonea, Approximation of Baskakov type Polya-Durrmeyer operators, Appl. Math. Comput., 2017(294):319-331.
[7] R. Bojanic and F. Cheng, Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation, J. Math. Anal. Appl.,1989, 141(1):136-151.
[8] X M. Zeng and F. Cheng, On the rate of approximation of Bernstein type operators, J. Approx. Theory, 2001, 109(2):242-256.
[9] B Y. Lian, Rate of approximation of bounded variation functions by the Bézier variant of Chlodowsky operators, J.Math.Inequal, 2013, 7(4):647-657.