Approximation properties of some Lupas-Durrmeyer type operators

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Abstract. By using Bojanic-Cheng’s method and analysis techniques, the authors study the rate of convergence of Lupas-Durrmeyer type operators for some absolutely continuous functions having a derivative equivalent to a bounded variation.

1 Introduction

In [1], Aral introduced some Lupas-Durrmeyer type operators

\[ L_n^{(1/n)}(f, x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_{0}^{1} p_{n,k}(t)f(t)dt, \]

where \( f \in C[0,1], p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!} \binom{n}{k}(nx)^k(n-nx)^{n-k}, \( n(k) = (n+1) \cdots (n+k-1), \)

\( p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \) and \( p_{n,k}^{(1/n)}(x) \) come from the density function of Polya distribution

\[ p_{n,k}^{(\alpha)}(x) = \frac{\prod_{v=0}^{k-1} (x+v\alpha) \prod_{\mu=0}^{n-k-1} (1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1} (1+\lambda\alpha)}, \] \( x \in [0,1]. \)

Recently, Gupta [2] introduced another Lupas-Durrmeyer type operators

\[ D_n^{(1/n)}(f, x) = n \sum_{k=1}^{n} p_{n,k}^{(1/n)}(x)\int_{0}^{1} p_{n-1,k-1}(t)f(t)dt + p_{n,0}^{(1/n)}(x)f(0), \]

and investigated the local and global approximation properties. Furthermore, the authors also considered the voronovskaya type asymptotic formula. Later, several researchers have made significant contributions in this direction. We refer the readers to some of the related papers [3-6].

The rate of approximation for functions with derivatives of bounded variation is an interesting topic. This is mainly originated from Bojanic-Cheng [7], then many scholars have done a lot of research in this field [8-9]. Since the introduction of the operators based on Polya distribution, the work related to this [3-6] has not stopped.

Inspired by this, this article studies the approximation of operator \( D_n^{(1/n)}(f, x) \) for some absolutely continuous functions \( DBV \), which having a derivative equivalent to a functions of...
bounded function $BV$.

We get some definition as follows.

**Definition 1**

$DBV[0,1]=\left\{ f \mid f(x) = f(0) + \int_0^1 h(t)dt \right\},$

where $x \in [0,1], h \in BV[0,1]$, i.e., $h$ is a function of bounded variation on $[0,1]$.

**Definition 2**

$$K_n(x,t) = n \sum_{k=1}^{n} p_{n,k}^{(1/n)}(x)p_{n-1,k-1}(t) + \delta(t),$$

where $\delta(t)$ is the Dirac delta function.

By the Lebesgue-Stieltjes integral representations, we have

$$D_n^{(1/n)}(f,x) = \int_0^1 f(t)K(x,t)dt.$$

### 2 Some lemmas

We start this section with the following useful lemmas, which will be used in the sequel.

**Lemma 1** (see [2]) For $e_i = t^i, i = 0, 1, 2$, we have

$$D_n^{(1/n)}(e_0, x) = 1, D_n^{(1/n)}(e_1, x) = \frac{nx}{n+1},$$

$$D_n^{(1/n)}(e_2, x) = \frac{n^2(n-1)x^2 + n(3n+1)x}{(n+1)^2(n+2)}.$$

**Remark 1** By simple applications of Lemma 1, we get

$$D_n^{(1/n)}(-x, x) = \frac{-x}{n+1},$$

$$D_n^{(1/n)}((t-x)^2, x) = \frac{(n+2-3n^2)x^2 + n(3n+1)x}{(n+1)^2(n+2)}.$$

**Remark 2** When $n$ sufficient large, we have

$$D_n^{(1/n)}((t-x)^2, x) \leq \frac{3\delta_n^2(x)}{n+1}.$$  \(\text{(4)}\)

where $\delta_n^2(x) = x(1-x) + \frac{1}{n+1}$.

**Lemma 2** When $n$ sufficient large, we have

$$D_n^{(1/n)}(|t-x|, x) \leq \sqrt{\frac{3}{n+1}}\delta_n(x).$$  \(\text{(5)}\)

**Proof.** By Cauchy-Schwarz inequality, we have

$$D_n^{(1/n)}(|t-x|, x) \leq \sqrt{D_n^{(1/n)}((t-x)^2, x) \cdot D_n^{(1/n)}(1, x)} \leq \sqrt{\frac{3}{n+1}}\delta_n(x).$$

The last inequality is obtained by Lemma 1 and remark 2.
Lemma 3 (i) For $0 \leq y < x < 1$, when $n$ sufficient large, there holds
\[
R_n(x, y) = \int_y^0 K_n(x, t) dt \leq \frac{3\delta_n^2(x)}{(n+1)(x-y)^2}.
\]
(ii) For $0 < x < z \leq 1$, when $n$ sufficient large, there holds
\[
1 - R_n(x, z) = \int_z^1 K_n(x, t) dt \leq \frac{3\delta_n^2(x)}{(n+1)(z-x)^2}.
\]

Proof. (i) By (3) and (4), we get
\[
R_n(x, y) = \int_y^0 K_n(x, t) dt \leq \int_0^y \left( \frac{x-t}{x-y} \right)^2 K_n(x, t) dt \leq \frac{1}{(x-y)^2} \int_0^1 (t-x)^2 K_n(x, t) dt
\]
\[
= \frac{1}{(x-y)^2} D_n^{(1/n)}((t-x)^2, x) \leq \frac{3\delta_n^2(x)}{(n+1)(x-y)^2}.
\]
(ii) Using a similar method, we get (ii) easily.

3 Conclusion

Theorem  Let $f \in DBV[0, 1]$. If $h(x+), h(x-)$ exist at a fixed point $x \in (0, 1)$, when $n$ sufficient large, then we have
\[
\left| D_n^{(1/n)}(f, x) - f(x) + \frac{\delta h(x) + h(x-)}{2(n+1)} \right| \leq \left| h(x+) - h(x-) \right| \sqrt{\frac{3}{4(n+1)}} \delta_n(x)
\]
\[
+ \frac{6\delta_n^2(x)}{(n+1)x(1-x)} \sum_{k=1}^{n} \frac{n+1-x}{x^k} V_{\varphi_n}(x) + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{n+1-x}{x^k} V_{\varphi_n}(x).
\]
Where
\[
\varphi_n(t) = \begin{cases} h(t) - h(x+), & x < t \leq 1; \\ 0, & t = x; \\ h(t) - h(x-), & 0 \leq t < x. \end{cases}
\]

Proof. Let $f$ satisfy the conditions of Theorem, by using Bojanic-Cheng’s method [7], we have
\[
f(t) = f(x) \int_x^t h(u),
\]
and $h(u)$ can be expressed as
\[
h(u) = \frac{h(x+) + h(x-)}{2} + \varphi_n(u) + \frac{h(x+) - h(x-)}{2} \text{sign}(u-x) + \delta_n(u) \left[ h(x) - \frac{h(x+) + h(x-)}{2} \right],
\]
where
\[
\delta_n(u) = \begin{cases} 
1, & u = x \\
0, & u \neq x 
\end{cases}, \quad \text{sign}(x) = \begin{cases} 
1, & x > 0; \\
0, & x = 0; \\
-1, & x < 0. 
\end{cases}
\]

From (6) and (7), and noting \( \int_{\mathbb{R}} \text{sign}(u-x) du = |x| \), \( \int_{\mathbb{R}} \delta_n(u) du = 0 \), we find that

\[
D_n^{(1/n)}(f, x) - f(x) = D_n^{(1/n)}(f(t) - f(x), x) = D_n^{(1/n)}(\int_x^t h(u) du, x)
\]

\[
= \frac{h(x)+h(x-)}{2} D_n^{(1/n)}(t-x, x) + \frac{h(x)-h(x-)}{2} D_n^{(1/n)}(|x|, x) + D_n^{(1/n)}(\int_x^t \varphi(u) du, x).
\]

By Remark 1, Remark 2 and Lemma 2, we have

\[
\left| D_n^{(1/n)}(f, x) - f(x) + \frac{x[h(x)+h(x-)]}{2(n+1)} \right| \leq \frac{|h(x)+h(x-)|}{2} D_n^{(1/n)}(t-x, x) + D_n^{(1/n)}(\int_x^t \varphi(u) du, x)
\]

\[
\leq |h(x)+h(x-)| \sqrt{\frac{3}{4(n+1)}} \delta_n(x) + D_n^{(1/n)}(\int_x^t \varphi(u) du, x). \quad (8)
\]

To complete the proof, we must estimate the term \( D_n^{(1/n)}(\int_x^t \varphi(u) du, x) \).

From (3), the term \( D_n^{(1/n)}(\int_x^t \varphi(u) du, x) \) can be stated as

\[
D_n^{(1/n)}(\int_x^t \varphi(u) du, x) = \int_0^t \left( \int_x^t \varphi(u) du \right) K_n(x, t) dt = \int_0^t \left( \int_x^t \varphi(u) du \right) dR_n(x, t)
\]

\[
= \int_0^t \left( \int_x^t \varphi(u) du \right) dR_n(x, t) + \int_x^t \left( \int_0^t \varphi(u) du \right) dR_n(x, t) + \int_0^t \left( \int_0^t \varphi(u) du \right) dR_n(x, t).
\]

Let \( \Delta_{1n}(f, x) = \int_0^t \left( \int_x^t \varphi(u) du \right) dR_n(x, t) + \Delta_{2n}(f, x) = \int_0^t \left( \int_x^t \varphi(u) du \right) dR_n(x, t) \), then we have

\[
D_n^{(1/n)}(\int_x^t \varphi(u) du, x) = \Delta_{1n}(f, x) + \Delta_{2n}(f, x). \quad (9)
\]

Fistly, we estimate \( \Delta_{1n}(f, x) \). Using partial integration and noticing \( R_n(x, 0) = 0, \int_x^t \varphi(u) du = 0 \), we get

\[
\Delta_{1n}(f, x) = R_n(x, t) \int_x^t \varphi(u) du |_0^t - \int_0^t R_n(x, t) \varphi(t) dt = -\int_0^t R_n(x, t) \varphi(t) dt
\]

\[
= - \left( \int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x \right) R_n(x, t) \varphi(t) dt.
\]

Thus, it follows that

\[
|\Delta_{1n}(f, x)| \leq \int_0^{x-x/\sqrt{n}} R_n(x, t) \varphi(t) dt + \int_{x-x/\sqrt{n}}^x R_n(x, t) \varphi(t) dt.
\]

From Lemma 3(i) and \( 0 \leq R_n(x, t) \leq 1 \), we have

\[
|\Delta_{1n}(f, x)| \leq \frac{3\delta_n^2(x)}{n+1} \int_0^{x-x/\sqrt{n}} \frac{\varphi(t)}{(x-t)^2} dt + \frac{\varphi}{\sqrt{n}} \frac{\varphi}{\sqrt{n}}.
\]

(10)
Putting \( t = x - \frac{x}{u} \) for the integral of (10), we get

\[
\int_0^x \frac{V(u)}{V(x-t)^2} \frac{dt}{u} = \frac{1}{x} \int_0^{\frac{x}{u}} V(u) du \leq \frac{2}{x \sum_{k=1}^{x} V(\phi_k)}.
\]  

(11)

From (10) and (11), it follows that

\[
|\Delta_n(f,x)| \leq \frac{6\sigma^2_n(x)}{(n+1)x} \sum_{k=1}^{x} \frac{V(x)}{V(\phi_k)} + \frac{x}{\sqrt{n}} \frac{V(x)}{V(\phi_k)}.
\]  

(12)

Using the same method, we get

\[
|\Delta_n(f,x)| \leq \frac{6\sigma^2_n(x)}{(n+1)(1-x)} \sum_{k=1}^{x} \frac{V(x)}{V(\phi_k)} + \frac{1-x}{\sqrt{n}} \frac{V(x)}{V(\phi_k)}.
\]  

(13)

Theorem now follows from (8), (9), (12) and (13). This completes the proof.

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