An approximate method for solving complex Ginzburg-Landau equation

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Abstract. The first integral method is applied to solve complex Ginzburg-Landau equation in this work. The evolution solutions for the equation are obtained. This method is based on the theory of commutative algebra, which can be applied to nonintegrable equations as well as to integrable ones. The first integral method supplied an effcient way to obtain traveling wave solutions of some nonlinear partial differential equations. This approach can also be applied to other nonlinear fractional differential equations.

Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of science. The generalized complex Ginzburg-Landau equation is one of the most-studied nonlinear equations in the physics community, which arises from a wide variety of fields, such as quantum field theory, superconductivity, superfluidity, Bose-Einstein condensation, liquid crystals and strings in field theory[1, 2, 3]. In optics, this equation (or a generalization of it) describes the essential features of processes in lasers[4, 5, 6, 7, 8]. Some of the numerical techniques designed under the non-standard methodologies have been popularized in the literature: tanh-sech method[9, 10, 11], Jacobi elliptic function expansion method[12], hyperbolic function method[13], Riccati equation method[14], F-expansion method[15]. The problem of finding first integrals of ordinary differential equations (ODEs) was initially considered by Darboux and by Lie.[16]

The first integral method is based on the ring theory of commutative algebra[17], which is widely used by many such as in [18,19] and by the reference therein and this approach can also be applied to other nonlinear fractional differential equations. We applied the first integral method for the complex Ginzburg-Landau equation and the evolution solutions

The first-integral method

For a given nonlinear evolution equation with two variables in the form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{xxx}, ...) = 0, \tag{1}$$

where $u = u(x; t)$, P is a polynomial u and its derivatives. Using a wave variable $\xi = x + \lambda t$, Eq.(1) is changed into ordinary differential equation, which can be rewritten as

$$Q(u, u', u'', u''', ...) = 0. \tag{2}$$

The prime in Eq.(2) denotes the derivative with respect to the same variable ξ. The equation $u(x, t) = f(\xi)$ is supposed to be the solutions of Eq.(2). Then, a new independent variable is read as the following Eq.(3), which leads a system of nonlinear ordinary differential equations Eq.(4). With the

$$X(\xi) = f(\xi); \quad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{3}$$

$$Y = \frac{\partial X(\xi)}{\partial \xi}; \quad \frac{\partial Y(\xi)}{\partial \xi} = F(X(\xi), Y(\xi)) \tag{4}$$

same conditions, the general solutions to Eq.(4) can be obtained directly by employing the qualitative theory of ordinary differential equations [17]. However, in general, it is really difficult for us to realize

this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for us to tell what these first integrals are. Fortunately, the first integral for Eq.(4) is obtained by employed the Division Theorem, which can help us obtain one first integral for Eq.(4) which reduces Eq. (2) to a firstorder integrable ordinary differential equation. The evolution solution for Eq.(1) is finally obtained by solving the first-order integrable ordinary differential equation.

The Division theorem for two variables in the complex domain **C** is read as: ***Division theorem***: *Suppose that G1[w; z], G[w; z] are polynomials in C[w; z] and G1[w; z] is irreducible in C[w; z]. If G[w; z] vanishes at all zero points of G1[w; z], then there exists a polynomial G2[w; z] in C[w; z] such that*

$$G[w; z] = G1[w; z]G2[w; z] \tag{5}$$

Complex Ginzburg-Landau Equation

The The complex Ginzburg-Landau equation, which is one of the most-studied nonlinear equations in the physics community, is studied in this section. The complex Ginzburg-Landau equation is read as:

$$u_t = u - (1 + ib)|u|^2 u + (1 + ia)u_{xx} \tag{6}$$

where a and b are constant parameters, u is the function of x and t. By using the transformation $u(x; t) = e^{\theta} f(\xi)$, $\xi = x + \lambda t$ and $\theta = \alpha x + \beta t$, where λ, α, and β are constants, the ordinary differential equation transferred from Eq.(6) is obtained

$$f_{\xi\xi} + \frac{2i\alpha - 2\alpha a - \lambda}{1 + ia} f_{\xi} + \frac{1 - \alpha^2 - ia\alpha^2 - i\beta}{1 + ia} f - \frac{1 + ib}{1 + ia} f^3 = 0. \tag{7}$$

With the help of Eq.(3), we obtain

$$\dot{X} = Y; \qquad \dot{Y} = \frac{1 + ib}{1 + ia}X^3 - \frac{2i\alpha - 2\alpha a - \lambda}{1 + ia}Y - \frac{1 - \alpha^2 - ia\alpha^2 - i\beta}{1 + ia}X. \tag{8}$$

According to the first-integral method and Division theorem, we suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of Eq. (8). $G(X; Y)$ can be expressed as $G(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0.$ The above expression is an irreducible polynomial in the complex domain $C[X; Y]$, then the above equation with parameter ξ is rewritten as

$$G(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0 \tag{9}$$

where $a_i(X(\xi))$, $(i = 0; 1; 2; :::m)$ are polynomials of X and $a_m(X(\xi)) \neq 0$. Eq.(9) is called the first integral for Eq.(8). According to the Division Theorem and the derivative of G, a polynomial $g(X)+h(X)Y$ in the complex domain $C[X; Y]$ must meet the relationship

$$\frac{dG}{d\xi} = \frac{dG}{dX}\frac{dX}{d\xi} + \frac{dG}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^{m} a_i(X)Y^i \tag{10}$$

Here, two different cases, $m = 1$ and $m = 2$ are assumed in Eq.(9) in this example.

Case A. Expanding Eq.(10) and equating the coefficients of $Y^i, (i = 0; 1; 2)$ on both side of with the parameter $m = 1$, we have Eq.(11,12,13)

$$\dot{a_1}(X) = h(X)a_1(X) \tag{11}$$

$$\dot{a_0}(X) = g(X)a_1(X) + h(X)a_0(X) - a_1 \frac{2i\alpha - 2\alpha a - \lambda}{1 + ia} \tag{12}$$

$$g(X)a_0(X) = a_1(X)(\frac{1 + ib}{1 + ia}X^3 - \frac{1 - \alpha^2 - ia\alpha^2 - i\beta}{1 + ia}X); \tag{13}$$

From Eq.(9), we known that $a_i(X)$ $(i = 0, 1)$ are polynomials, then $a_1(X)$ is constant and $h(X) = 0$ are deduced from Eq.(11). For simplicity, $a_1(X)$ take the form $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X)) = 1$ only. Suppose that $g(X) = A_0 + A_1X$ with parameter A_0 and A_1 to be determined, then we find $a_0(X)$ read as

$$a_0(X) = \frac{1}{2}A_1X^2 + A_0X - \frac{2i\alpha - 2\alpha a - \lambda}{1 + ia} + R, \tag{14}$$

where R is some an integration constant. Substituting a_1, g and a_0 into Eq.(13) and setting all the coefficients of powers X to be zero, then we obtain a system of algebraic equations and by solving it, we obtain

$$A_1 = \sqrt{\frac{2(1 + ib)}{1 + ia}}; \quad A_0 = \frac{2(2i\alpha - 2\alpha a - \lambda)}{3(1 + ia)}; \quad R = 0 \tag{15}$$

where λ, b, α, a and $a \neq i$ are constants. Using the conditions Eq.(15) and Eq.(9), we obtain the evolution solution of the Eq.(6)

$$\lambda = \frac{2 - 2i\beta - 2\alpha^2 - 2\alpha^2 a^2}{a\alpha}; \quad M_1 = \frac{2i(x + \lambda t)\alpha}{3(1 + ia)} + 4a\alpha\sqrt{1 + ia} + 2\lambda\sqrt{1 + ia} \quad M_2 = \frac{2a(x + \lambda t)\alpha}{3(1 + ia)} + 4i\alpha\sqrt{1 + ia} + \frac{(x + \lambda t)\lambda}{3(1 + ia)}$$

$$f(x, t) = -\frac{2e^{M_1}\sqrt{1 + ia}(2i\alpha - 2a\alpha - \lambda)}{e^{M_2} - 3\sqrt{2(1 + ib)}e^{M_1}(1 + ia)} \qquad u(x, t) = e^{\alpha x + \beta t}f(x, t). \tag{16}$$

Case B. Equating the coefficients of $Y_i,(i = 0; 1; 2; 3)$ on both side of Eq.(10) with $m = 2$, we obtain

$$a_2\dot{}(X) = h(X)a_2(X) \tag{17}$$

$$a_1\dot{}(X) = h(X)a_1(X) + g(X)a_2(X) - 2a_2\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia} \tag{18}$$

$$a_0\dot{}(X) = g(X)a_1(X) + h(X)a_0(X) - \frac{a_1(2i\alpha - 2a\alpha - \lambda)}{1 + ia} - 2a_2(\frac{1 + ib}{1 + ia}X^3 - \frac{1 - \alpha^2 - ia\alpha^2 - i\beta}{1 + ia}X) \tag{19}$$

$$g(X)a_0(X) = a_1(X)(\frac{1 + ib}{1 + ia}X^3 - \frac{1 - \alpha^2 - ia\alpha^2 - i\beta}{1 + ia}X); \tag{20}$$

Since $a_i(X)$ $(i = 0,1, 2)$ are polynomials, $a_2(X)$ is constant and $h(X) = 0$ are yielded from Eq.(17), we set $a_2 = 1$ as usual. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $deg(g(X)) = 1$, therefore we have: $g(X) = A_1X + A_0$ is given. $a_1(X)$ is obtained from Eq.(18) as follow

$$a_1(X) = \frac{1}{2}A_1X^2 + A_0X - 2\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia}X + R; \tag{21}$$

where R is an integration constant.

$$a_0(X) = \frac{A_1^2X^4}{8} + \frac{(3A_0A_1 - 5A_1\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia})X^3}{6} - R(\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia} - A_0)X + R_0 + (A_0^2 + 2(\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia})^2 - 3A_0(\frac{2i\alpha - 2a\alpha - \lambda}{1 + ia}) + RA_1)\frac{X^2}{2} \tag{22}$$

Substituting a_2, a_1, a_0, g, into Eq.(9) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = \frac{1}{2}\sqrt{\frac{1 + ib}{1 + ia}}; \quad A_0 = \frac{8(2i\alpha - 2a\alpha - \lambda)}{9(1 + ia)}; \quad R = R_0 = 0 \tag{23}$$

$$A_1 = -\frac{1}{2}\sqrt{\frac{1 + ib}{1 + ia}}; \quad A_0 = \frac{8(2i\alpha - 2a\alpha - \lambda)}{9(1 + ia)}; \quad R = R_0 = 0 \tag{24}$$

where $a \neq i$, b, and α are constants. Using the conditions Eq.(22)(23) and Eq.(9), we obtain the traveling wave solution of the Eq.(6)

$$\lambda = \frac{2 - 2i\beta - 2\alpha^2 - 2\alpha^2 a^2}{a\alpha}, \quad M = \frac{1 + ib}{1 + ia} \quad f(x, t) = -\frac{8e^{\frac{iM^{1/4}(x + t\lambda)}{2} + 2\sqrt{2}}}{2\sqrt{M}e^{4\sqrt{2} + iM^{1/4}(x + t\lambda)} - 1}; \quad f(x, t) = -\frac{8e^{\frac{iM^{1/4}(x + t\lambda)}{2} + 2\sqrt{2}}}{2\sqrt{M}e^{4\sqrt{2}} - e^{iM^{1/4}(x + t\lambda)}}; \tag{25}$$

$$u(x, t) = e^{\alpha x + \beta t}f(x, t). \tag{26}$$

$$f(x, t) = -\frac{2\sqrt{2}i}{\sqrt{M}}coth[\frac{-M^{1/4}(x + t\lambda) + 4\sqrt{2}i}{2}]\sqrt{\sqrt{M} - \sqrt{M}tanh[\frac{-M^{1/4}(x + t\lambda) + 4\sqrt{2}i}{2}]^2};$$

$$f(x, t) = \frac{2\sqrt{2}i}{\sqrt{M}}coth[\frac{M^{1/4}(x + t\lambda) + 4\sqrt{2}i}{2}]\sqrt{\sqrt{M} - \sqrt{M}tanh[\frac{M^{1/4}(x + t\lambda) + 4\sqrt{2}i}{2}]^2}; \tag{27}$$

$$u(x, t) = e^{\alpha x + \beta t}f(x, t). \tag{28}$$

Summary

The evolution solutions for thecomplex Ginzburg-Landau equation are obtained by using the first integral method, which is based on the theory of commutative algebra. The simulation figures are

shown with the corresponding parameters. The first integral method, which can be applied to nonintegrable equations as well as to integrable ones, is a standard, direct and computerizable method.

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