

# The traveling wave solution of (3+1) dimensional Kadomtsev-Petviashvili equation by using the firstintegral method

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**Abstract.** The first integral method, based on the theory of commutative algebra, is an efficient method for obtaining traveling wave solutions of some nonlinear partial differential equations, which is applied to solve (3+1) dimensional Kadomtsev-Petviashvili equation in this work. The traveling wave solutions for the equation are obtained. This method can be applied to nonintegrable equations as well as to integrable ones.

## Introduction

During the past three decades, nonlinear partial differential equations, which are widely used to describe complex phenomena in various fields of science, had recently proved to be valuable tools to the modeling of many physical phenomena, and have gained the focus of many studies due to their frequent appearance in various applications such as fluid flow, signal processing, control theory, systems identification, finance, fractional dynamics, and other areas. Many powerful methods for obtaining traveling wave solutions of nonlinear partial differential equation have been presented, hyperbolic function method[1], F-expansion method[2], Jacobi elliptic function expansion method[3], Riccati equation method[4], tanh-sech method[5, 6, 7]. The first integral method was first proposed for solving Burger-KdV equation [8, 9] which is based on the ring theory of commutative algebra[10]. The useful first integral method is widely used by many such as in [11, 12, 13, 14, 15, 16, 17, 18] and by the reference therein. The (3+1) dimensional Kadomtsev-Petviashvili equation is studied in this work, which was first written in 1970 by Soviet physicists Boris B. Kadomtsev and Vladimir I. Petviashvili. The equation, came as a natural generalization of the KdV equation, can be used to model water waves of long wavelength with weakly non-linear restoring forces and frequency dispersion and can also be used to model waves in ferromagnetic media, as well as two-dimensional matter-wave pulses in Bose-Einstein condensates[19].

## The first-integral method

For a given nonlinear partial differential equation in the form

$$P(u, ut, ux, utt, uxx, uxt, uxxx, \dots) = 0,$$

(1)

Using a wave variable  $\xi = k(x+ly+ pz-\lambda t)$ , Eq.(1) is changed into ordinary differential equation, which can be rewritten as

$$Q(u, u', u'', u''', \dots) = 0, \quad (2)$$

The prime in Eq.(2) denotes the derivative with respect to the same variable  $\xi$ . The equation  $f(\xi) = u(x, t)$  is supposed to be the solutions of Eq.(2). Then, a new independent variable is introduced as the following

$$X(\xi) = f(\xi); \quad Y = f\xi(\xi),$$

(3)

which leads a system of nonlinear ordinary differential equations

$$Y = X\xi(\xi); \quad Y\xi = F(X(\xi); Y(\xi))$$

(4)

By employing the qualitative theory of ordinary differential equations [10]. With the same conditions, the general solutions to Eq.(4) can be obtained directly if we can find the integrals for Eq. (4). Generally, it is really difficult for us to realize this result even for the first integral, for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals. The Division Theorem can help us obtain one first integral for Eq.(4) which reduces Eq. (2) to a first-order integrable ordinary differential equation. The traveling wave solution for Eq.(1) is obtained by solving the first-order integrable ordinary differential equation. The Division theorem for two variables in the complex domain  $\mathbf{C}$  is read as:

*Division theorem: Suppose that  $G1[w; z]$ ,  $G[w; z]$  are polynomials in  $C[w; z]$  and  $G1[w; z]$  is irreducible in  $C[w; z]$ . If  $G[w; z]$  vanishes at all zero points of  $G1[w; z]$ , then there exists a polynomial  $G2[w; z]$  in  $C[w; z]$  such that*

$$G[w; z] = G1[w; z]G2[w; z] \quad (5)$$

### The (3+1) dimensional Kadomtsev-Petviashvili equation

The (3+1) dimensional Kadomtsev-Petviashvili equation[20, 21, 22] is studied in this section:

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0 \quad (6)$$

By using the transformation  $\xi = k(x + ly + pz - \lambda t)$ , where  $k, l, p$  and  $\lambda$  are constants, and integrating the Eq. (6) with respect to  $\xi$  and taking the integration constant as  $R$  yields the ordinary differential equation

$$u_{\xi\xi} + \frac{3u^2}{k^2} + \frac{-\lambda \pm 3l^2 \pm 3p^2}{k^2}u - \frac{R}{k^2} = 0 \quad (7)$$

Using Eq.(3), we obtain

$$\dot{X} = Y; \quad \dot{Y} = \frac{R}{k^2} - \frac{3u^2}{k^2} - \frac{-\lambda \pm 3l^2 \pm 3p^2}{k^2}u; \quad (8)$$

According to the first-integral method,  $X(\xi)$  and  $Y(\xi)$  are supposed to be the nontrivial solutions of

Eq. (8), we obtain  $G(X, Y) = \sum_{i=0}^m a_i(X)Y^i = 0$  which is an irreducible polynomial in the complex domain  $C[X; Y]$ , then the above equation is rewritten as

$$G(X(\xi), Y(\xi)) = \sum_{i=0}^m a_i(X(\xi))Y^i(\xi) = 0 \quad (9)$$

where  $a_i(X(\xi))$ , ( $i = 0; 1; 2; \dots; m$ ) are polynomials of  $X$  and  $a_m(X(\xi)) \neq 0$ . Eq.(9) is called the first integral for Eq.(8). According to the Division Theorem, a polynomial  $g(X) + h(X)Y$  in the complex domain  $C[X; Y]$  must meet the relationship

$$\frac{dG}{d\xi} = \frac{dG}{dX} \frac{dX}{d\xi} + \frac{dG}{dY} \frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^m a_i(X)Y^i \quad (10)$$

Here, we assume that two different cases,  $m = 1$  and  $m = 2$  in Eq.(9) in this example.

Case I. Equating the coefficients of  $Y_i$ , ( $i = 0, 1, 2$ ) on both side of Eq.(10) with  $m = 1$ , we obtain

$$a_1 \dot{X} = h(X)a_1(X) \quad (11)$$

$$a_0 \dot{X} = g(X)a_1(X) + h(X)a_0(X) \quad (12)$$

$$g(X)a_0(X) = a_1(X) \left( \frac{R}{k^2} - \frac{3X^2}{k^2} - \frac{-\lambda \pm 3l^2 \pm 3p^2}{k^2}X \right); \quad (13)$$

As known  $a_i(X)$  ( $i = 0; 1$ ) are polynomials, then  $a_1(X)$  is constant and  $h(X) = 0$  are obtained from Eq.(11). Usually,  $a_1(X) = 1$ . Balancing the degrees of  $g(X)$  and  $a_0(X)$ , we conclude that  $\deg(g(X)) = 1$  only.

$$a_0 = \frac{1}{2}A_1X^2 + A_0X + B_0 \quad (14)$$

is obtained with the condition  $g(X) = A_0 + A_1X$ , where  $B_0$  is some an integration constant.

Substituting  $a_1, g, a_0$ , into Eq.(13) and setting all the coefficients of powers  $X$  to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = 0; \quad A_0 = \pm \frac{\sqrt{\lambda \mp 3l^2 \mp 3p^2}}{k}; \quad B_0 = \frac{R}{k^2A_0} \quad (15)$$

where  $k \neq 0$ ,  $\lambda$ ,  $l$ , and  $p$  are constants. Using the conditions Eq.(15) and Eq.(8), we obtain the traveling wave solution of the Eq.(6)

$$u(x, y, z, t) = -\frac{R}{A_0^2 k^2} + C_0 e^{-A_0 k(x+ly+pz-\lambda t)}, \quad (16)$$

where  $C_0$  is a constant.

Case II. Equating the coefficients of  $Y_i$  ( $i = 0, 1, 2, 3$ ) on both side of Eq.(10) with  $m = 2$ , we obtain

$$a_2'(X) = h(X)a_2(X) \quad (17)$$

$$a_1'(X) = h(X)a_1(X) + g(X)a_2(X) \quad (18)$$

$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X) - 2a_2\left(\frac{R}{k^2} - \frac{3X^2}{k^2} - \frac{-\lambda \pm 3l^2 \pm 3p^2}{k^2}X\right) \quad (19)$$

$$g(X)a_0(X) = a_1(X)\left(\frac{R}{k^2} - \frac{3X^2}{k^2} - \frac{-\lambda \pm 3l^2 \pm 3p^2}{k^2}X\right); \quad (20)$$

Since  $a_i(X)$  ( $i = 0, 1, 2$ ) are polynomials,  $a_2(X)$  is constant and  $h(X) = 0$  are yielded from Eq.(17), we set  $a_2 = 1$  as usual. Balancing the degrees of  $g(X)$ ,  $a_1(X)$  and  $a_2(X)$ , we conclude that  $\deg(g(X)) = 0$ , therefore we have:  $g(X) = A_1$  is given.  $a_1(X)$  is obtained from Eq.(18) as follow

$$a_1(X) = A_1 X + A_0; \quad (21)$$

where  $A_0$  is an integration constant.

$$a_0(X) = \frac{A_1^2 X^2 + 2A_1 A_0 X}{2} - \frac{2RX - 2X^3 + (\lambda \mp 3l^2 \mp 3p^2)X^2}{k^2}; \quad (22)$$

Substituting  $a_2$ ,  $a_1$ ,  $a_0$ ,  $g$ , into Eq.(13) and setting all the coefficients of powers  $X$  to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain  $A_1 = 0$ ;  $A_0 = 0$  where  $k \neq 0$ ,  $\lambda$ ,  $l$ , and  $p$  are constants. Using the conditions  $A_1 = 0$ ;  $A_0 = 0$  and Eq.(9), we obtain the traveling wave solution of the Eq.(6)

$$u(x, y, z, t) = \frac{\lambda - 3l^2 - 3p^2}{2}(1 + \tan\phi^2); \quad (23)$$

where  $\phi = \frac{\sqrt{3l^2 - \lambda + 3p^2}}{2}(x + ly + pz - \lambda t)$ .

## Conclusions

The travelling wave solutions for the Eq.(6) are obtained by using the first integral method. People can extract important information from the singularity when studying the water waves of long wavelength with weakly non-linear restoring forces and frequency dispersion and model waves in ferromagnetic media. In all, the first integral method is a standard, direct and computerizable method. The performance of this method is reliable and effective, especially for the nonlinear equation with complicated and tedious algebraic calculation.

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