

Bayesian Estimation of the Scale Parameter of the Marshall-Olkin Exponential Distribution under Progressively Type-II Censored Samples

Mukhtar M. Salah

Basic Engineering Science, College of Engineering, Majmaah University
Majmaah, Kingdom of Saudi Arabia
m.salah@mu.edu.sa

Abstract:

This paper studies the Bayes estimator, the maximum likelihood estimator and the approximate likelihood estimator of the scale parameter for the Marshall-Olkin exponential distribution under the progressive type-II censored sample. All the estimators, Bayes estimator, maximum likelihood estimator and approximate likelihood estimator are presented and derived in simple forms. It observed that the Bayes estimator and the maximum likelihood estimator can not be solved analytically, hence it is solved numerically. Finally the comparison method is presented in order to compare the performance between these estimators.

Keywords: Progressive censoring; approximate maximum likelihood estimator; Bayes estimator; exponential distribution.

2000 Mathematics Subject Classification: 62G30, 62E99, 60E05, 62H10, 62F10.

Received 14 March 2017

Accepted 5 December 2017

1 Introduction

Let Z be a random variable from the Marshall-Olkin exponential distribution (MOE) distribution with the scale parameter λ and shape parameter α . The probability density function (pdf) of Z is given as follow

$$f(z) = \frac{\alpha e^{-\frac{z}{\lambda}}}{\lambda (1 - (1 - \alpha)e^{-\frac{z}{\lambda}})^2}, \quad z \geq 0, \alpha > 0 \text{ and } \lambda > 0. \quad (1.1)$$

and its cumulative distribution function (cdf) is given as

$$F(z) = 1 - \frac{\alpha e^{-\frac{z}{\lambda}}}{(1 - (1 - \alpha)e^{-\frac{z}{\lambda}})}, \quad z \geq 0, \alpha > 0 \text{ and } \lambda > 0. \quad (1.2)$$

The pdf and cdf of standard MOE distribution are given respectively as follows:

$$f(x) = \frac{\alpha e^{-x}}{(1 - (1 - \alpha)e^{-x})^2}, \quad 0 \leq x < \infty, \alpha > 0. \quad (1.3)$$

$$F(x) = 1 - \frac{\alpha e^{-x}}{(1 - (1 - \alpha)e^{-x})}, \quad 0 \leq x < \infty, \alpha > 0. \quad (1.4)$$

where $X = \frac{Z}{\lambda}$. Note that when $\alpha = 1$, in Eqs. (1.3) and (1.4) then the MOE distribution reduces to the standard exponential distribution, and when $\alpha = 2$, the MOE distribution reduces to the half logistic distribution. The hazard rate $h(x)$ for the MOE distribution, is given by

$$h(x) = \frac{1}{1 - (1 - \alpha)e^{-x}}, \quad x \geq 0, \alpha > 0.$$

[6] showed that when $\alpha \geq 1$ then the hazard rate $h(x)$ is increasing and if $0 < \alpha < 1$, $h(x)$ is decreasing. So the family of MOE distributions is an increasing failure rate (IFR) family when $\alpha \geq 1$ and a decreasing failure rate (DFR) family when $0 < \alpha < 1$. For more details see [10] and [12].

Censored sampling arises in a life-testing experiment when ever the experimenter doesn't observe (either intentionally or unintentionally) the failure times of all units placed on a life-test. For example consider a life-testing experiment where n items are kept under observation, these items could be systems, computers, individuals in a clinical trial, in reliability study experiment, so that the removal of units from the experimentation is pre-planned and intentional, and is done in order to provide saving in terms of time and cost associated with testing. The data obtained from such experiments are called censored data. There are many types of censoring scheme, here we mention some of them, let us consider n units are placed on a life-test then, type-I (time) censoring: Suppose it is decided to terminate the experiment at a pre-determined time t , so that only failure time of these items that failed prior to this time recorded, the data so obtained from this process constitute a type-I censored sample. Type-II censoring: If the experiment is terminated at the r th failure, that is at time $X_{r:n}$, we obtain type-II censored sample, here r is fixed, while $X_{r:n}$ the duration of the experiment is random. Many articles in this literature have discussed inferential method under type-I and type-II censoring for various parametric families of distributions, for more details, see for example, [1, 3, 5, 7, 9, 11].

A generalization of type-II censored sample is a progressive type-II censoring: Suppose n units taken from the same population are placed on a life test. At the first failure time of one of the n units, a number R_1 of the surviving units is randomly withdrawn from the test, at the second failure time, another R_2 surviving units are selected at random and taken out of the experiment, and so on. Finally at the m th failure, the remaining $R_m = n - m - R_1 - R_2 - \dots - R_{m-1}$ unit are removed. In this scheme (R_1, R_2, \dots, R_m) is pre-fixed. The resulting m order failure times, which denote by

$$X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, X_{2:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)},$$

are referred to as progressive type-II censored order statistics. The special case when $R_1 = R_2 = \dots = R_{m-1} = 0$, so that $R_m = n - m$ this scheme reduces to the

conventional type-II censoring scheme, also when $R_1 = R_2 = \dots = R_m = 0$, so that $m = n$, then no censoring happen (complete data case). For more details discussion about progressive censoring, one may refer to [2]. If the failure times are based on an absolutely continuous distribution function F with probability density function (pdf) f , the joint probability density function of the progressive censored failure times $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$, is given by

$$f_{X_{1:m:n}, \dots, X_{m:m:n}}(x_1, x_2, \dots, x_m) = A(n, m-1) \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{R_i},$$

$$-\infty < x_1 < x_2 < \dots < x_m < \infty, \quad (1.5)$$

where $f(\cdot)$ and $F(\cdot)$ are, respectively, pdf and (cdf) of the random sample and

$$A(n, m-1) = n(n-1-R_1)(n-2-R_1-R_2)\dots(n-m+1-R_1-\dots-R_{m-1}).$$

The rest of the paper is organized as follows. In Section 2, approximate maximum likelihood estimator (AMLE) of the scale parameter λ of MOE distribution is presented and used as an initial starting points to find the maximum likelihood estimator (MLE) of λ . In Section 3, Bayes estimator of λ is studied and presented. Finally, in Section 4, numerical computations and calculations are presented to compare between these estimators.

2 Approximate Maximum Likelihood Estimation

In this section, we derive the AMLEs of the scale parameters λ of the MOE distribution under progressively type-II censored sample by using [2] algorithm. Let $Y_{1:m:n}, Y_{2:m:n}, \dots, Y_{m:m:n}$ denote a progressively type-II censored sample from MOE distribution with pdf and cdf as in Eqs. (1.1) and (1.2) respectively. Let $X_i = Y_i/\lambda$, $i = 1, 2, \dots, m$, then X_i 's are simply order statistics from a sample of size n from the standard MOE distribution. One can approximate the function $F(x_i)$ by expanding it in a Taylor series around the point $E(X_{i:m:n}) = \nu_{i:m:n}$.

It is known that

$$F(X_{i:m:n}) \stackrel{D}{=} U_{i:m:n},$$

where $U_{i:m:n}$ is the i th progressively type-II censored order statistic from uniform $U(0, 1)$ distribution. We then have

$$X_{i:m:n} \stackrel{D}{=} F^{-1}(U_{i:m:n}),$$

with

$$F^{-1}(u) = \ln \left(\frac{1 - (1 - \alpha)u}{1 - u} \right).$$

Consequently,

$$\begin{aligned}\nu_{i:m:n} &= E(X_{i:m:n}) \approx F^{-1}(\gamma_{i:m:n}) \\ &\approx \ln \left(\frac{1 - (1 - \alpha) \gamma_{i:m:n}}{1 - \gamma_{i:m:n}} \right),\end{aligned}$$

where $\gamma_{i:m:n} = E(U_{i:m:n})$. From [2], it is known that

$$\gamma_{i:m:n} = 1 - \prod_{j=m-i+1}^m \frac{j + R_{m-j+1} + \dots + R_m}{j + 1 + R_{m-j+1} + \dots + R_m}, \quad i = 1, 2, \dots, m. \quad (2.1)$$

By expanding $F(x_i)$ using Taylor series expansion around the point $\nu_{i:m:n}$ and keeping only the first two terms for approximation we get

$$\begin{aligned}F(X_i) &\approx F(\nu_{i:m:n}) + (x_i - \nu_{i:m:n}) f(\nu_{i:m:n}), \\ &= w_i + \delta_i x_i,\end{aligned} \quad (2.2)$$

where

$$w_i = F(\nu_{i:m:n}) - \nu_{i:m:n} f(\nu_{i:m:n}), \quad i = 1, 2, \dots, m$$

and

$$\delta_i = f(\nu_{i:m:n}), \quad i = 1, 2, \dots, m.$$

by using Eq. (1.5), one can find the MLE of λ by differentiating Eq.(2.3) with respect to λ and then solving Eq.(2.4) numerically.

$$\ln L = \ln c - m \ln \alpha - m \ln \lambda + \sum_{i=1}^m (R_i + 1) \ln [1 - F(X_i)] + \sum_{i=1}^m \ln [1 - (1 - \alpha) F(X_i)] \quad (2.3)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{-1}{\alpha \lambda} \left[\alpha m - \sum_{i=1}^m (R_i + 2 - \alpha) X_i + \sum_{i=1}^m (R_i + 2)(1 - \alpha) F(X_i) X_i \right] = 0. \quad (2.4)$$

Upon using Eq. (2.4) and (2.2), the AMLE of λ based on the progressively type-II censored sample can be obtained by solving Eq. (2.5)

$$\frac{\partial \ln L}{\partial \lambda} = \frac{-1}{\alpha \lambda} \left[\alpha m - \sum_{i=1}^m (R_i + 2 - \alpha) X_i + \sum_{i=1}^m (R_i + 2)(1 - \alpha) X_i (w_i + \delta_i X_i) \right] = 0, \quad (2.5)$$

after simplifying Eq. (2.5), we get

$$A\lambda^2 + B\lambda + C = 0. \quad (2.6)$$

By solving the quadratic Eq. (2.6) with respect to λ , we obtain the AMLE of λ as

$$\hat{\lambda} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad (2.7)$$

where

$$\begin{aligned} A &= \alpha m, \\ B &= \sum_{i=1}^m [(R_i + 2)(1 - \alpha) w_i - (R_i + 2 - \alpha)] y_i \\ C &= \sum_{i=1}^m (R_i + 2)(1 - \alpha) y_i^2 \delta_i. \end{aligned}$$

The AMLEs of scale parameters of MOE distribution could be used as starting points of the numerical solution in Newton-Raphson method of Eq. (2.4) to find the MLE of λ .

3 Bayes Estimation

In this section, we present the derivation for the Bayes estimator for the scale parameter λ of the MOE distribution. To see this, let $Z_1 \leq Z_2 \leq \dots \leq Z_m$ be a progressively type-II censored sample from MOE distribution with pdf and cdf as in Eqs. (1.1) and (1.2) respectively. Let us consider the natural conjugate family of the prior distribution for parameter λ as follow:

$$\pi(\lambda) \propto \left(\frac{1}{\lambda}\right)^{a+1} e^{-\frac{b}{\lambda}}, \lambda > 0, a > 0 \text{ and } b > 0. \quad (3.1)$$

The posterior density of λ is given by combining Eq. (1.5) with Eq. (3.1) as

$$\pi(\lambda|Z) \propto \left(\frac{1}{\lambda}\right)^{m+a+1} e^{-\frac{1}{\lambda} \left[\sum_{i=1}^m (R_i+1)Z_i + b \right]} \prod_{i=1}^m \left(1 - (1 - \alpha)e^{-\frac{Z_i}{\lambda}}\right)^{-(R_i+2)}. \quad (3.2)$$

The cdf of the Bayes estimator of λ under the square error loss (SEL) is the posterior mean and given by

$$\hat{\lambda}_B = \frac{\int_0^\infty \lambda \pi(\lambda|Z) d\lambda}{\int_0^\infty \pi(\lambda|Z) d\lambda}. \quad (3.3)$$

To find the Bayes estimator of λ by numerical integration method, we use Eq. (3.2) and (3.3) which is due to the complex form of the likelihood function. To obtain the Bayes estimator of λ , on can use Eq. (3.3) as given in Eq. (3.4)

$$\widehat{\lambda}_B = E(\lambda|Z) = \frac{E^* \left[\lambda \prod_{i=1}^m \left(1 - (1 - \alpha) e^{-\frac{Z_i}{\lambda}} \right)^{-(R_i+2)} \right]}{E^* \left[\prod_{i=1}^m \left(1 - (1 - \alpha) e^{-\frac{Z_i}{\lambda}} \right)^{-(R_i+2)} \right]}, \quad (3.4)$$

where E^* denote the expectation with respect to inverse gamma distribution. Since Eq.(3.4) can not be solved analytically, we use an approximation method for [13] to find the numerical approximate solution. To do this, we assume

$$k(\lambda) = \frac{\partial \ln \pi(\lambda|Z)}{\partial \lambda} = -(m + a + 1) \lambda + \sum_{i=1}^m (R_i + 1) Z_i + b - \sum_{i=1}^m \frac{(1 - \alpha) (R_i - 2) Z_i e^{-\frac{Z_i}{\lambda}}}{\left(1 - (1 - \alpha) e^{-\frac{Z_i}{\lambda}} \right)}. \quad (3.5)$$

From Eq. (3.5) it follows that $\hat{\lambda}^*$ is only mode of the posterior density in Eq.(3.2) for simplicity let

$$k(\lambda) = \Phi(\lambda) + \Psi(\lambda),$$

where

$$\Phi(\lambda) = -(m + a + 1) \lambda + \sum_{i=1}^m (R_i + 1) Z_i + b,$$

and

$$\Psi(\lambda) = \sum_{i=1}^m \frac{(1 - \alpha) (R_i - 2) Z_i e^{-\frac{Z_i}{\lambda}}}{\left(1 - (1 - \alpha) e^{-\frac{Z_i}{\lambda}} \right)}.$$

Since $\Phi(\lambda)$ and $\Psi(\lambda)$ are decreasing and increasing in $(0, \infty)$ respectively. Therefore Eq.(3.5) admits a unique solution for $\hat{\lambda}^*$.

Let $L(\lambda; z)$ be likelihood function of λ based on n observations and $\pi(\lambda|z)$ denote the posterior distribution of λ . Then posterior mean of $\Phi(\lambda)$ is given by

$$E[\Phi(\lambda)|Z] = \int \Phi(\lambda) \pi(\lambda|z) d\lambda = \frac{\int e^{nL^*(\lambda)} d\lambda}{\int e^{nL(\lambda)} d\lambda}, \quad (3.6)$$

where

$$L(\lambda) = \frac{1}{n} \ln \pi(\lambda|z) \quad (3.7)$$

and

$$L^*(\lambda) = \pi(\lambda) + \frac{1}{n} \ln \pi(\lambda). \quad (3.8)$$

Following [13], the Eq. (3.6) can be approximated as follow

$$\begin{aligned} E[\Phi(\lambda)|Z] &= \left(\frac{|\zeta^*|}{|\zeta|}\right)^{0.5} e^{\left(n\left[L^*(\hat{\lambda}^*)-L(\hat{\lambda})\right]\right)}, \\ &= \left(\frac{|\zeta^*|}{|\zeta|}\right)^{0.5} \frac{\Phi(\hat{\lambda}^*)\pi(\hat{\lambda}^*|z)}{\pi(\hat{\lambda}|z)}. \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \frac{\partial^2 L(\lambda)}{\partial \lambda^2} &= \frac{-2}{n\lambda^3} \left[-(m+a+1)\lambda + \sum_{i=1}^m (R_i+1)z_i + b + \sum_{i=1}^m \frac{(R_i+2)(1-\alpha)z_i e^{-\frac{z_i}{\lambda}}}{\left(1-(1-\alpha)e^{-\frac{z_i}{\lambda}}\right)} \right] + \\ &\quad \frac{1}{n\lambda^2} \left[-(m+a+1) + \sum_{i=1}^m \frac{(R_i+2)(1-\alpha)z_i^2 e^{-\frac{z_i}{\lambda}}}{\lambda^2 \left(1-(1-\alpha)e^{-\frac{z_i}{\lambda}}\right)^2} \right], \\ \frac{\partial^2 L^*(\lambda)}{\partial \lambda^2} &= \frac{\partial^2 L(\lambda)}{\partial \lambda^2} - \frac{1}{n} \frac{(m+a+1)^2}{\left(-(m+a+1)\lambda + \sum_{i=1}^m (R_i+1)z_i + b \right)^2}. \end{aligned}$$

where $\hat{\lambda}^*$ and $\hat{\lambda}$ maximize $L^*(\lambda)$ and $L(\lambda)$ respectively. ζ^* and ζ are minus the inverse of the second derivatives of $L^*(\lambda)$ and $L(\lambda)$ at $\hat{\lambda}^*$ and $\hat{\lambda}$ respectively.

We applying this approximation to get the Bayes estimator of the scale parameter λ as follow

$$L(\lambda) = \frac{1}{n} \left[-(m+a+1) \ln \lambda - \frac{1}{\lambda} \left(\sum_{i=1}^m (R_i+1)z_i + b \right) - \sum_{i=1}^m (R_i+2) \ln \left(1 - (1-\alpha)e^{-\frac{z_i}{\lambda}} \right) \right]. \quad (3.10)$$

and

$$L^*(\lambda) = L(\lambda) + \frac{1}{n} \ln \lambda \quad (3.11)$$

By substituting Eq. (3.10) and (3.11) in (3.9), the Bayes estimator $\hat{\lambda}_{AB}$ of

a function $\Phi(\lambda) = \lambda$ under the SEL takes of the form

$$\begin{aligned} \hat{\lambda}_{AB} = E[\lambda|Z] &= \left\{ \left(\frac{|\zeta^*|}{|\zeta|} \right)^{0.5} \left(\frac{\lambda}{\lambda^*} \right)^{\frac{m+a+1}{m+a+1}} e^{\left(\sum_{i=1}^m (R_i+1)Z_i + b \right) \left(\frac{1}{\lambda} - \frac{1}{\lambda^*} \right)} \right. \\ &\quad \left. \times \prod_{i=1}^m \frac{\left(1 - (1-\alpha)e^{-\frac{Z_i}{\lambda}} \right)^{R_i+2}}{\left(1 - (1-\alpha)e^{-\frac{Z_i}{\lambda^*}} \right)^{R_i+2}} \right\}. \end{aligned} \quad (3.12)$$

4 Numerical Computation

In this section, we present a simulation study and numerical computations to compare the performances of the different estimators, the AMLE's and Bayes estimator with the MLE's of λ . To this end, by using the algorithm presented by [4], we generate progressively type-II censored samples from the standard MOE distribution where $\lambda = 1$. We compute the AMLE from Eqs. (2.7). The MLE's of λ are obtained by solving the nonlinear Eq. (2.4) in which the AMLE was used as a starting values for Newton-Raphson method. The Bayes estimators of λ are obtained by solving Eq. (3.12). All the computation are computed using Mathematica 6.0.1 software package over 4000 Monte Carlo simulations. The simulations are carried out for sample sizes $n = 10, 20, 30, 50$. Different choices of the effective sample size m , and different progressive censoring schemes are considered. For simplicity in notation, we have used the same notation as in [8], as $((m-1)*0, n-m)$ and $(n-m, (m-1)*0)$ respectively; for example $(5, 4*0)$ denotes the progressive censoring scheme $(5, 0, 0, 0, 0)$.

We present the results for the AMLEs, the MLEs and Bayes estimator when $\lambda = 1$ for some fixed shape parameter $\alpha = 1.5, 2, 2.5, 3$. in Table(1-4).

Finally, we present here an example for simulated data from MOE distribution to see the performance of the different estimators of the scale parameter λ from the MOE distribution.

Example 4.1

A progressively type-II censored sample of size $m = 10$ and a complete sample size $n = 31$ from MOE distribution with $\lambda = 2$ and censoring scheme $(1, 2, 3, 4, 5, 0, 0, 0, 0, 6)$ was generating using Balakrishnan and Sandhu(1995) algorithm. The generated progressively type-II sample is

$$\{0.321312, 0.352673, 0.838508, 1.57235, 1.5746, 2.07522, 2.20029, 3.348, 4.32915, 4.36173\}$$

it found that AMLE is 1.9938, the MLE is 1.9986 and finally Bayes estimator is 1.8041. Its observed that the MLE is the closest estimator to the scale parameter $\lambda = 2$ but the Bayes estimator is slightly far from $\lambda = 2$.

Table 1: The AMLE's , MLE's and Bayes estimators of the scale parameter λ when the data are simulated from MOE distribution with $\alpha = 1.5$

n	m	Scheme	AMLE	MLE	Bayes
10	5	(4 * 0, 5)	1.2613	1.0459	0.9037
		(5, 4 * 0)	1.3789	0.9651	0.7424
		(1, 1, 1, 1, 1)	1.3124	1.0200	1.1789
20	5	(4 * 0, 15)	2.4123	0.9965	0.9616
		(15, 4 * 0)	2.5789	0.9265	1.0005
		(5, 0, 5, 0, 5)	2.4488	1.0008	0.9475
	10	(9 * 0, 10)	1.2522	1.0555	0.9104
		(10, 9 * 0)	1.3471	0.9873	1.0004
		4 * 0, 5, 5, 4 * 0)	1.3099	1.0511	0.9063
	15	(5 * 1, 10 * 0)	0.9673	0.9708	0.8797
		(10 * 0, 5 * 1)	0.9083	0.9205	0.8353
		(5, 14 * 0)	0.9679	0.9678	0.9063
		(14 * 0, 5)	0.9574	1.0461	0.9417
	30	(20, 9 * 0)	1.9691	1.0521	0.9167
		(9 * 0, 20)	1.8055	1.0757	0.9214
		(3 * 0, 5, 5, 5, 5, 3 * 0))	1.8864	1.0247	0.8802
30	15	(15, 14 * 0)	1.3798	1.0824	0.9785
		(14 * 0, 15)	1.2456	1.0465	0.9446
	20	(10, 19 * 0)	1.0537	0.9748	0.9517
		(19 * 0, 10)	1.0067	1.0165	0.9422
	25	(5, 24 * 0)	0.9152	1.0231	0.9616
		(24 * 0, 5)	0.9078	1.0516	0.9889
50	20	(30, 19 * 0)	1.6283	0.9669	0.8964
		(19 * 0, 30)	1.4904	0.9639	0.8908
	30	(20, 29 * 0)	1.1372	0.9776	0.9283
		(29 * 0, 20)	1.0618	0.9774	0.9281

Table 2: The AMLE's , MLE's and Bayes estimators of the scale parameter λ when the data are simulated from MOE distribution with $\alpha = 2$

n	m	Scheme	AMLE	MLE	Bayes
10	5	(4 * 0, 5)	0.9386	0.9020	0.9452
		(5, 4 * 0)	1.0951	0.9099	0.9574
		(1, 1, 1, 1, 1)	1.0868	1.0538	0.8930
20	5	(4 * 0, 15)	1.6860	1.0651	1.0956
		(15, 4 * 0)	1.990	1.0275	1.2417
		(5, 0, 5, 0, 5)	1.7447	1.0792	1.0861
	10	(9 * 0, 10)	0.9941	1.0232	0.9013
		(10, 9 * 0)	1.1896	1.0490	0.9239
		(4 * 0, 5, 5, 4 * 0)	1.1396	1.0864	0.9503
	15	(5 * 1, 10 * 0)	0.9061	1.0133	0.9269
		(10 * 0, 5 * 1)	0.9233	1.0780	0.9888
		(5, 14 * 0)	0.9795	1.0834	0.9923
		(14 * 0, 5)	0.8994	1.0383	0.9544
	30	(20, 9 * 0)	1.5942	1.1073	0.9745
		(9 * 0, 20)	1.2445	1.0301	0.9788
		(3 * 0, 5, 5, 5, 5, 3 * 0))	1.4291	1.0639	0.9290
30	15	(15, 14 * 0)	1.1135	0.9660	0.9409
		(14 * 0, 15)	0.9927	1.0389	0.9505
	20	(10, 19 * 0)	0.9603	1.0162	0.9499
		(19 * 0, 10)	0.8905	1.0176	0.9534
	25	(5, 24 * 0)	0.8628	1.0114	0.9583
		(24 * 0, 5)	0.8800	1.0548	0.9527
50	20	(30, 19 * 0)	1.3174	1.0124	0.9467
		(19 * 0, 30)	1.1037	1.0036	0.9358
	30	(20, 29 * 0)	0.9920	0.9813	0.9386
		(29 * 0, 20)	0.8799	0.9625	0.9208

Table 3: The AMLE's , MLE's and Bayes estimators of the scale parameter λ when the data are simulated from MOE distribution with $\alpha = 2.5$

n	m	Scheme	AMLE	MLE	Bayes
10	5	(4 * 0, 5)	0.9035	0.9951	0.8130
		(5, 4 * 0)	1.1074	0.9838	0.8435
		(1, 1, 1, 1, 1)	1.0596	1.0848	0.8867
20	5	(4 * 0, 15)	1.2635	0.9471	0.9439
		(15, 4 * 0)	1.6357	0.9897	0.9587
		(5, 0, 5, 0, 5)	1.4150	1.0985	0.9753
	10	(9 * 0, 10)	0.9181	1.0214	0.9126
		(10, 9 * 0)	1.1171	1.0365	0.9253
		(4 * 0, 5, 5, 4 * 0)	1.0253	1.0547	0.9399
	15	(5 * 1, 10 * 0)	0.9514	1.0522	0.9724
		(10 * 0, 5 * 1)	0.8945	1.0167	0.9426
		(5, 14 * 0)	0.9223	0.9991	0.9262
		(14 * 0, 5)	0.8543	0.9922	0.9210
	30	(20, 9 * 0)	1.3269	0.9923	0.9816
		(9 * 0, 20)	1.0632	1.0455	0.9238
		(3 * 0, 5, 5, 5, 5, 3 * 0))	1.1806	0.9642	0.9403
30	15	(15, 14 * 0)	1.0496	0.9737	0.9022
		(14 * 0, 15)	0.9034	1.0077	0.9322
	20	(10, 19 * 0)	0.9484	1.0021	0.9450
		(19 * 0, 10)	0.8769	1.0077	0.9520
	25	(5, 24 * 0)	0.9086	1.0429	0.9939
		(24 * 0, 5)	0.8777	1.0294	0.9829
50	20	(30, 19 * 0)	1.1446	1.0026	0.9448
		(19 * 0, 30)	0.9577	1.0185	0.9563
	30	(20, 29 * 0)	0.9853	1.0269	0.9868
		(29 * 0, 20)	0.8892	1.0225	0.9829

Table 4: The AMLE's , MLE's and Bayes estimators of the scale parameter λ when the data are simulated from MOE distribution with $\alpha = 3$

n	m	Scheme	AMLE	MLE	Bayes
10	5	(4 * 0, 5)	0.9059	0.9966	0.8345
		(5, 4 * 0)	1.0794	0.9807	0.8140
		(1, 1, 1, 1, 1)	0.9935	1.0159	0.8480
20	5	(4 * 0, 15)	1.0716	0.9929	0.8014
		(15, 4 * 0)	1.5046	1.0134	0.8309
		(5, 0, 5, 0, 5)	1.2137	1.0464	0.8563
	10	(9 * 0, 10)	0.9093	1.0437	0.9406
		(10, 9 * 0)	1.0836	1.0194	0.9181
		(4 * 0, 5, 5, 4 * 0)	1.0389	1.0694	0.9616
	15	(5 * 1, 10 * 0)	0.9782	1.0469	0.9757
		(10 * 0, 5 * 1)	0.9318	1.0303	0.9623
		(5, 14 * 0)	0.9382	0.9909	0.9248
		(14 * 0, 5)	0.9218	1.0253	0.9589
30	10	(20, 9 * 0)	1.3086	1.0526	0.9552
		(9 * 0, 20)	0.9542	1.0331	0.9225
		(3 * 0, 5, 5, 5, 5, 3 * 0))	1.1440	1.0475	0.9331
	15	(15, 14 * 0)	1.0708	1.0157	0.9468
		(14 * 0, 15)	0.9047	1.0250	0.9551
	20	(10, 19 * 0)	0.9276	0.9700	0.9202
		(19 * 0, 10)	0.8942	1.0069	0.9566
	25	(5, 24 * 0)	0.9218	1.0246	0.9809
		(24 * 0, 5)	0.9069	1.0215	0.9799
50	20	(30, 19 * 0)	1.1299	0.9929	0.9411
		(19 * 0, 30)	0.8886	0.9968	0.9417
	30	(20, 29 * 0)	0.9618	0.9897	0.9545
		(29 * 0, 20)	0.8584	0.9802	0.9462

5 Conclusion

This paper studied the estimators of the unknown scale parameter λ under progressively type-II censored samples from the MOE distribution. It is observed that the MLE and Bayes estimators cannot be solved analytically. The AMLE is used as starting point in finding the MLE and [13] method is used to find the numerical approximate solution of Bayes estimator. It found that the performance of the MLE and the AMLE are very closed to each other but the Bayes estimator is slightly far from the MLE and AMLE.

Acknowledgement

The author would like to thank the Deanship of Scientific Research at Majmaah University for supporting this work . Also the author would like to thank the editors for their cooperation and grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

- [1] N. Balakrishnan, Order Statistics from the Half Logistic Distribution, *Journal of Statistical Computation and Simulation* **20** (1985) 287-309.
- [2] N. Balakrishnan, and R. Aggarwala, Progressive Censoring: Theory, Method and Applications, Birkhauser, Boston (2000).
- [3] N. Balakrishnan, and C. Cohen, Order Statistics and Inference: Estimation Methods, Academic Press, Boston (1991).
- [4] N. Balakrishnan, and R. A. Sandhu, A Simple Simulation Algorithm for Generating Progressive Type-II Censored Samples, *The American Statistician* **49** (1995) 229-230.
- [5] N. Balakrishnan, E. Cramer, U. Kamps, and N. Schenk, Progressive Type-II Censored Order Statistics from Exponential Distributions, *Statistics* **35** (2001) 537-556.
- [6] M. E. Ghitany, E. K. Al-Hussaini, and R. A. A-Jarallah, Marshall-Olkin Extended Weibull Distribution and its Application to Censored Data, *Journal of Applied Statistics* **32(10)** (2005) 1025-1034.

- [7] C. Kim, and K. Han, Estimation of the scale parameter of the half-logistic distribution under progressive type-II censored sample, *Stat Paper* **51** (2010) 375-387.
- [8] H. K. T. Ng, P. S. Chan, and N. Balakrishnan, Estimation of Parameters from Progressively Censored Data Using EM Algorithm, *Computational Statistics and Data Analysis* **39** (2002) 371-386.
- [9] B. Pradhan, and D. Kundu, On Progressively Censored Generalized Exponential Distribution, *Test* **18** (2009) 497-515.
- [10] M. M. Salah, On Marshall-Olkin Exponential Order Statistics and Associated Inferences, Ph.D Thesis, The University of Jordan, Amman, Jordan (2010).
- [11] M. M. Salah, Moments From Progressive Type-II Censored Data Of Marshall-Olkin Exponential Distribution, *International Journal of Applied Mathematical Research*, **1** (4) (2012) 771-786.
- [12] M. M. Salah, M. Z. Raqab, and M. Ahsanullah, Marshall-Olkin Exponential Distribution: Moments of Order Statistics, *Journal of Applied Statistical Science* **17**(1) (2009) 81-92.
- [13] L. Tierney, and J. Kadane, Accurate approximations for posterior moments and marginal densities. *Journal of American Statistical Association* **81** (393) (1986) 82-86.