

A Hesitant Fuzzy Multiple Attribute Decision Making Method Based on Complementary Judgment Matrix

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Abstract—Hesitant fuzzy set allow an element in the domain with a set of membership degree between 0 and 1, which is a powerful tool to express the uncertainty alternative assessment value of decision makers. A hesitant fuzzy multiple attribute decision making method is proposed based on additive consistency theory of fuzzy complementary judgment matrix. First, we define mean residual between two hesitant fuzzy elements and a new fuzzy complementary judgment matrix and then prove it has additive consistency; on this basis, we gain a new weighted fuzzy complementary judgment matrix. Then, the fuzzy complementary judgment matrix is implemented to rank and select alternatives. Finally, an example is given to illustrate the feasibility and effectiveness of the proposed method.

Keywords—hesitant fuzzy set; fuzzy complementary judgment matrix; additive consistency; multiple attribute decision making

I. INTRODUCTION

Since the fuzzy sets proposed by Zadeh [1], it has been brought into focus on dealing with uncertainty problem. Then, some extension of fuzzy sets have been introduced because of the complexity and uncertainty of real life. Thereinto, type-2 fuzzy set that allows the membership of a given element as a fuzzy set proposed by Zadeh [2]; Dubois and Prade [3] presented n-type fuzzy set [3] which generalized 2-type fuzzy set; fuzzy multiple set introduced by Yager [4], in which the elements can be repeated more than once; Atanassov [5] presented the concept of intuitionistic fuzzy set (IFS) and analyzed the fuzzy set by constructed membership function, non-membership function and hesitancy function.

In the process of the ideal decision making, rational individual thinking and tacit understanding group thinking are the key to form consistent accuracy judgment. However, human limited knowledge and limited logos make it hard for experts to give a precise, satisfy consistency conditions, the overall optimal judgment information, always hesitate in several possible values [6]. As an extension of fuzzy set, Torra [7] proposed the concept of hesitant fuzzy set in which the membership degree of an element in the domain has a set of possible values between 0 and 1. Proposed concept is a powerful tool to deal imprecise and uncertain decision making problem. Currently, hesitant fuzzy multiple attribute decision making problem has caused wide attention of scholars both at home and abroad and has achieved phased research results. Many kinds of hesitant fuzzy aggregation operators [8, 9, 10], distance measures and similarity measures [11, 12, 13] have been studied to deal with multiple attribute decision making problem. Xia et al. [14] proposed a set of hesitant fuzzy

information fusion operators and discussed their generalizations. Xu et al. [15, 16] introduced some basic concepts of hesitant fuzzy set such as correlation coefficient, entropy, cross entropy and so on and applied them to the multiple attribute decision making. Wei [17] presented a set of hesitant fuzzy prioritized operators in multiple attribute decision making. Xia et al. [18] and Zhang [19] researched the application of hesitant fuzzy aggregation operator in multiple attribute decision making. Farhadinia [20, 21] proposed the formula of distance, similarity degree, entropy on hesitant fuzzy set and a new method for ranking alternatives expressed with hesitant fuzzy set. The consistency of the preference relationship will have a direct effect on the rationality of the final ranking of the alternatives, and the lack of consistency will produce the result that is not logical. Recently, increasing scholars have come to research the multiple attribute decision making problem which has consistency. Zhang et al. [22] develop a consistency-improving process to adjust unacceptably multiplicative consistency into an acceptably multiplicative one. Zhu [23] develop some consistency measures for HFLPRs to ensure that the DMs are being neither random nor illogical. Zhang et al. [24] introduce the acceptable consistency of the PLPR and use aggregation operators to obtain the comprehensive preference values of alternatives and then rank the alternatives. Tan et al. [25] constructed an optimization model based on the maximization of group consistency index to research the multiple attribute decision making problem. Rank and select alternatives is the purpose of solving multiple attribute decision making problem. At present, The common methods including aggregation operators [26, 27, 28, 29], TOPSIS[30, 31, 32, 33], VIKOR[34, 35], grey relational analysis[36, 37] et al. methods. This paper research the additive consistency of hesitant fuzzy preference relation by constructed a hesitant fuzzy complementary judgment matrix and then proposed a new approach to analyze hesitant fuzzy multiple attribute decision making problem.

The remaining sections of this paper are set up as follows: section 2 reviews and fixes basic concepts; section 3 constructs a new fuzzy complementary judgment matrix and proves it has additive consistency and then it's generalization is generalized; section 4, a new approach to multiple attribute decision making under hesitant fuzzy environment is developed; section 5 provides a numerical example to demonstrate the feasibility and validity of the proposed method and in section 6 conclusions are gives.

II. PRELIMINARIES

In this section, we will give some basic definitions of hesitant fuzzy set, fuzzy complementary judgment matrix, fuzzy complementary judgment matrix with additive consistency.

A. Hesitant Fuzzy Set

Limited knowledge and logos of people make it hard for experts to give a consistency condition. Based on this, Torra [7] proposed the concept of hesitant fuzzy set as follow:

Definition 1. [7] let X be a reference set. An hesitant fuzzy set (HFS) E on X is defined in terms of a function $h_E(x)$ that returns a subset of $[0, 1]$ when applied to X .

For convenience, the HFS can be expressed by Wei [17] mathematically as follow:

$$E = \{ \langle x, h_E(x) \rangle \mid x \in X \} \quad (1)$$

where $h_E(x)$ is a set of values in $[0, 1]$, which represent the possible membership degrees of the element $x \in X$ to E . For convenience, we call $h_E(x)$ a hesitant fuzzy element (HFE).

Example 1. Let $X = \{x_1, x_2, x_3\}$ is the reference set, respectively, suppose that $h_E(x_1) = \{0.2, 0.8\}$, $h_E(x_2) = \{0.4, 0.5, 0.7\}$ and $h_E(x_3) = \{0.4, 0.6, 0.8\}$ are the HFEs of $x_i (i=1, 2, 3)$ for a set E . Then E can be considered as a HFS, i.e.

$$E = \{ \langle x_1, \{0.2, 0.8\} \rangle, \langle x_2, \{0.4, 0.5, 0.7\} \rangle, \langle x_3, \{0.4, 0.6, 0.8\} \rangle \}. \quad (2)$$

B. Fuzzy Preference Relation

Definition 2. [38] Let $R = (r_{ij})_{n \times n}$ be a judgment matrix, then $R = (r_{ij})_{n \times n}$ is fuzzy complementary judgment matrix if for any $i, j \in I$ satisfy

$$r_{ij} \geq 0, r_{ij} + r_{ji} = 1, r_{ii} = 0.5 \quad (2)$$

Definition 3. [39] Let $R = (r_{ij})_{n \times n}$ be a fuzzy complementary judgment matrix, then $R = (r_{ij})_{n \times n}$ is fuzzy complementary judgment matrix with additive consistency if for any $i, j, k \in I$ satisfy

$$r_{ij} = r_{ik} + r_{kj} - 0.5 \quad (3)$$

Definition 4. [40] For any complementary judgment matrix $R = (r_{ij})_{n \times n}$, then $R = (r_{ij})_{n \times n}$ is fuzzy complementary judgment matrix with additive consistency if exist standardization $\beta = (\beta_1, \beta_2, \dots, \beta_n)^T$ and $c (c \geq \frac{n-1}{2})$ for any $i, j \in I$ satisfy

$$r_{ij} = c(\beta_i - \beta_j) + 0.5. \quad (4)$$

By the formula (4) above, we can obtain by calculation

$$\beta_j = \frac{1}{n} - \frac{1}{nc} \left(\sum_{i=1}^n r_{ij} - 0.5n \right). \quad (5)$$

III. HESITANT FUZZY PREFERENCE RELATION AND IT'S GENERALIZATION

In this section, we will give definitions of mean residual and priority degree and prove it has additive consistency and then generalize its general model.

Definition 5. [41] Supposed two HFES $h_i = \{u_{ik}\}, i = 1, 2, \dots, n, k = 1, 2, \dots, l_i$; $h_q = \{v_{qd}\}, q = 1, 2, \dots, n, d = 1, 2, \dots, l_q$. Let

$$G_{iq} = \{ (u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} > 0, (u_{ik}, v_{qd}) \in h_i \times h_q \} \quad (6)$$

$$\bar{G}_{iq} = \{ (u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} < 0, (u_{ik}, v_{qd}) \in h_i \times h_q \} \quad (7)$$

$$G_{iq0} = \{ (u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} = 0, (u_{ik}, v_{qd}) \in h_i \times h_q \} \quad (8)$$

Mean residual is defined as follow:

$$D_{iq} = \begin{cases} \frac{1}{l_i l_q} \sum_{(u_{ik}, v_{qd}) \in G_{iq}} (u_{ik} - v_{qd}), & G_{iq} \neq \phi \\ 0, & G_{iq} = \phi \end{cases} \quad (9)$$

$$\bar{D}_{iq} = \begin{cases} \frac{1}{l_i l_q} \sum_{(u_{ik}, v_{qd}) \in \bar{G}_{iq}} (v_{qd} - u_{ik}), & \bar{G}_{iq} \neq \phi \\ 0, & \bar{G}_{iq} = \phi \end{cases} \quad (10)$$

Where D_{iq} denotes the average residual amount for h_i f h_q ; \bar{D}_{iq} denotes the average residual amount for h_q f h_i .

Definition 6. Let D_{iq} denotes the average residual amount for h_i f h_q ; \bar{D}_{iq} denotes the average residual amount for h_q f h_i . The priority degree r_{iq} for h_i f h_q is defined by

$$r_{iq} = \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}|} (D_{iq} - D_{qi}) + 0.5 \quad (11)$$

Obviously, $D_{iq} = \bar{D}_{iq}$, r_{iq} denotes the degree of h_i f h_q in some extent. If $r_{iq} = 1$, denotes h_i over h_q completely; If $r_{iq} = 0.5$ denotes the priority degree for h_i over h_q is 0.5; $r_{iq} = 0$ denotes h_q over h_i completely.

Property 1. $R = (r_{iq})_{n \times n}$ is fuzzy complementary judgment matrix which has additive consistency.

Proof.

$$|D_{iq} - D_{qi}| \leq \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}| \quad (12)$$

then

$$\left| \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}|} (D_{iq} - D_{qi}) \right| \leq 0.5 \quad (13)$$

so $0 \leq r_{iq} \leq 1$.

Obviously,

$$r_{ii} = 0.5, r_{iq} + r_{qi} = 1 \quad (14)$$

so, $R = (r_{iq})_{n \times n}$ is fuzzy complementary judgment matrix.

Because,

$$D_{im} - D_{mi} = D_{im} - \bar{D}_{im} \quad (15)$$

$$= \frac{1}{l_i l_m} \left[\sum_{(u_{ik}, g_{mr}) \in G_{im}} (u_{ik} - g_{mr}) - \sum_{(g_{mr}, u_{ik}) \in \bar{G}_{im}} (g_{mr} - u_{ik}) \right] \quad (16)$$

$$+ \sum_{(u_{ik}, g_{mr}) \in G_{im0}} (u_{ik} - g_{mr}) \quad (17)$$

$$= \frac{1}{l_i l_m} \left[l_m \sum_{(u_{ik}, g_{mr}) \in G_{im}} u_{ik} - l_i \sum_{(u_{ik}, g_{mr}) \in G_{im}} g_{mr} \right] \quad (18)$$

$$- l_i \sum_{(u_{ik}, g_{mr}) \in \bar{G}_{im}} g_{mr} + l_m \sum_{(u_{ik}, g_{mr}) \in \bar{G}_{im}} u_{ik} \quad (19)$$

$$+ l_m \sum_{(u_{ik}, g_{mr}) \in G_{im0}} u_{ik} - l_i \sum_{(u_{ik}, g_{mr}) \in G_{im0}} g_{mr} \quad (20)$$

$$= \frac{1}{l_i} \sum_{k=1}^{l_i} u_{ik} - \frac{1}{l_m} \sum_{r=1}^{l_m} g_{mr} \quad (21)$$

Similarly,

$$D_{mq} - D_{qm} = \frac{1}{l_m} \sum_{r=1}^{l_m} g_{mr} - \frac{1}{l_q} \sum_{e=1}^{l_q} v_{qe} \quad (22)$$

Then,

$$r_{im} + r_{mq} = 0.5 \quad (23)$$

$$= \frac{1}{2 \max_{\forall i, m \in 1, 2, \dots, n} |D_{im} - D_{mi}|} (D_{im} - D_{mi}) \quad (24)$$

$$+ \frac{1}{2 \max_{\forall m, q \in 1, 2, \dots, n} |D_{mq} - D_{qm}|} (D_{mq} - D_{qm}) + 0.5 \quad (25)$$

$$= \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}|} (D_{im} - D_{mi} + D_{mq} - D_{qm}) + 0.5 \quad (26)$$

$$= \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}|} \left(\frac{1}{l_i} \sum_{k=1}^{l_i} u_{ik} - \frac{1}{l_m} \sum_{r=1}^{l_m} g_{mr} \right. \quad (27)$$

$$\left. + g_{mr} \frac{1}{l_m} \sum_{r=1}^{l_m} g_{mr} - \frac{1}{l_q} \sum_{e=1}^{l_q} v_{qe} \right) + 0.5 \quad (28)$$

$$= \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq} - D_{qi}|} (D_{iq} - D_{qi}) + 0.5 \quad (29)$$

$$= r_{iq} \quad (30)$$

So, the property 1 is proved.

Considering the complexity and uncertainty in the real world, we proposed the general model of hesitant fuzzy average residual.

Definition 7. Supposed two HFE_S

$$h_i = \{u_{ik}\}, i = 1, 2, \dots, n, d = 1, 2, \dots, l_i;$$

$$h_q = \{v_{qd}\}, q = 1, 2, \dots, n, d = 1, 2, \dots, l_q. \text{ Let}$$

$$G_{iq} = \{(u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} > 0, (u_{ik}, v_{qd}) \in h_i \times h_q\} \quad (31)$$

$$\bar{G}_{iq} = \{(u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} < 0, (u_{ik}, v_{qd}) \in h_i \times h_q\} \quad (32)$$

$$G_{iq0} = \{(u_{ik}, v_{qd}) \mid u_{ik} - v_{qd} = 0, (u_{ik}, v_{qd}) \in h_i \times h_q\} \quad (33)$$

General model average residual is defined as follow:

$$D_{iq}(\alpha) = \begin{cases} \frac{1}{l_i l_q} \sum_{(u_{ik}, v_{qd}) \in G_{iq}} (u_{ik}^\alpha - v_{qd}^\alpha), & G_{iq} \neq \emptyset \\ 0, & G_{iq} = \emptyset \end{cases} \quad (34)$$

$$\bar{D}_{iq}(\alpha) = \begin{cases} \frac{1}{l_i l_q} \sum_{(u_{ik}, v_{qd}) \in \bar{G}_{iq}} (v_{qd}^\alpha - u_{ik}^\alpha), & \bar{G}_{iq} \neq \emptyset \\ 0, & \bar{G}_{iq} = \emptyset \end{cases} \quad (35)$$

where $\alpha > 0$, $D_{iq}(\alpha)$ denote the average residual of h_i f h_q in general model; $\bar{D}_{iq}(\alpha)$ denote the average residual of h_q f h_i in general model.

Definition 8. Let $D_{iq}(\alpha)$ denotes the average residual amount for h_i f h_q in general model; $\bar{D}_{iq}(\alpha)$ denotes the average residual amount for h_q f h_i in general model. The priority degree $r_{iq}(\alpha)$ for h_i f h_q is defined by

$$r_{iq}(\alpha) = \frac{1}{2 \max_{\forall i, q \in 1, 2, \dots, n} |D_{iq}(\alpha) - D_{qi}(\alpha)|} (D_{iq}(\alpha) - D_{qi}(\alpha)) + 0.5 \quad (36)$$

where $\alpha > 0$, $D_{iq} = \bar{D}_{iq}$, $r_{iq}(\alpha)$ denote the priority degree of h_i f h_q of some extent in general model. If $r_{iq}(\alpha) = 1$, denotes h_i over h_q completely; If $r_{iq}(\alpha) = 0.5$, denotes the priority degree for h_i over h_q is 0.5; $r_{iq}(\alpha) = 0$ denotes h_q over h_i completely.

Example 2. Suppose that $X = \{x\}$. $H_1(x) = \{0.9, 0.5\}$, $H_2(x) = \{0.8, 0.7, 0.3\}$ and $H_3(x) = \{0.8, 0.5\}$ are three HFE_S on X .

From Definition 7, when $\alpha = 0.1$, we have

$$D_{12}(0.1) = 0.0309, D_{21}(0.1) = 0.0251. \quad (37)$$

$$D_{13}(0.1) = 0.0170, D_{31}(0.1) = 0.0112. \quad (38)$$

$$D_{23}(0.1) = 0.0128, D_{32}(0.1) = 0.0128. \quad (39)$$

From Definition 8, we have $r_{13}(0.1) = 0.6602$, $r_{23}(0.1) = 0.1602$.

Similarly,

when $\alpha = 1$, we have $r_{13}(1) = 0.75$, $r_{23}(1) = 0.25$.

when $\alpha = 10$, we have $r_{13}(10) = 0.9657$, $r_{23}(10) = 0.4657$.

when $\alpha = 20$, we have $r_{13}(20) = 0.9859$, $r_{23}(20) = 0.4859$.

From this example, we learn about α take different value, r_{iq} will change accordingly. When α is given a larger value, the value of r_{iq} will more and more closer 0, 0.5 or 1. If the value of r_{iq} is close to 0.5, we cannot know about the priority degrees between $H_1(x)$ and $H_2(x)$ clearly. Conversely, if the value of r_{iq} is close to 0 or 1, we can know about the priority degrees between $H_1(x)$ and $H_2(x)$ clearly. So the value of α will affect the order relations in some extent.

Property 2. $R(\alpha) = (r_{iq}(\alpha))_{n \times n}$ is fuzzy complementary judgment matrix which has additive consistency.

Proof. The proof is similar to Theorem 1 and thus omitted.

IV. A NEW APPROACH TO HESITANT FUZZY MULTIPLE ATTRIBUTE DECISION MAKING

For a hesitant fuzzy multiple attribute decision making problem, let $A = \{A_1, A_2, \dots, A_s\}$ be a set of alternatives and $C = \{C_1, C_2, \dots, C_t\}$ be a set of attributes. Where $\omega = [\omega_1, \omega_2, \dots, \omega_t]^T$ denotes the attribute weight vector satisfying $\sum_{j=1}^t \omega_j = 1$ and $\omega_j \in [0, 1]$, $j = 1, 2, \dots, t$. A group of expert provide possible membership degree of $A_i (i = 1, 2, \dots, s)$ on $C_j (j = 1, 2, \dots, t)$ and express as hesitant fuzzy element h_{ij} . The hesitant fuzzy decision matrix $H = (h_{ij})_{s \times t}$ of A on C is

$$H \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ M & M & M & M \\ h_{s1} & h_{s2} & \dots & h_{st} \end{bmatrix} \quad (40)$$

Definition 9. [42] For all of the $R_{C_j}(\alpha)$, a new hesitant fuzzy average operator is defined by

$$WA_M(R_{C_1}(\alpha), R_{C_2}(\alpha), \dots, R_{C_t}(\alpha)) = \sum_{j=1}^t \lambda_j R_{C_j}(\alpha). \quad (41)$$

Where, $R_{C_j}(\alpha) = (r_{iq}^{C_j}(\alpha))_{s \times s}$, $\alpha > 0$, $\lambda_j > 0$, $r_{iq}^{C_j}(\alpha)$ denote the priority degree of h_i f h_q for attribute C_j in general model.

Definition 10. [42]

Let

$$\bar{R}(\alpha) = (\bar{r}_{iq}(\alpha))_{s \times s} = \sum_{j=1}^t \omega_j R_{C_j}(\alpha) \quad (42)$$

$\bar{R}(\alpha)$ denote the fuzzy complementary judgment matrix of h_i f h_q on the basis of attribute weighting ω_j in general model. Where $\alpha > 0$, $\sum_{j=1}^t \omega_j = 1$ and $\omega_j \in [0, 1]$,

$R_{C_j}(\alpha) = (r_{iq}^{C_j}(\alpha))_{s \times s}$, $r_{iq}^{C_j}(\alpha)$ denote the priority degree of h_i f h_q for attribute C_j in general model.

Property 3. [42] Suppose $R(\alpha) = (r_{iq}(\alpha))_{n \times n}$ is fuzzy complementary judgment matrix which has additive consistency, then $\bar{R}(\alpha) = (\bar{r}_{iq}(\alpha))_{s \times s} = \sum_{j=1}^t \omega_j R_{C_j}(\alpha)$

satisfied $\sum_{j=1}^t \omega_j = 1$ and $\omega_j \in [0, 1]$, $j = 1, 2, \dots, t$ is also fuzzy complementary judgment matrix which has additive consistency.

Proof. From theorem 2, Definition 8 and Definition 9, we have

$$\bar{r}_{iq}(\alpha) = \sum_{j=1}^t \omega_j r_{iq}^{C_j}(\alpha) = \sum_{j=1}^t \omega_j (r_{ip}^{C_j}(\alpha) + r_{pq}^{C_j}(\alpha) - 0.5) \quad (43)$$

$$= \sum_{j=1}^t \omega_j r_{ip}^{C_j}(\alpha) + \sum_{j=1}^t \omega_j r_{pq}^{C_j}(\alpha) - 0.5 \quad (44)$$

$$= \bar{r}_{ip}(\alpha) + \bar{r}_{pq}(\alpha) - 0.5 \quad i, p, q = 1, 2, \dots, s \quad (45)$$

Obviously, $r_{ii} = 0.5$, $0 \leq \bar{r}_{iq}(\alpha) \leq 1$, $\bar{r}_{iq}(\alpha) + \bar{r}_{qi}(\alpha) = 1$.

So, the theorem is proved.

In the following, a new method based on the fuzzy complementary judgment matrix is proposed to solving the multiple attribute decision making problems, in which the information about attribute weights is given, and the attribute values take the form of hesitant fuzzy element, the new approach involves the following steps:

Step 1: Let $H = (h_{ij})_{s \times t}$ be a hesitant fuzzy decision-making matrix, where h_{ij} of each alternative $H_i \in H$ of each attribute $C_j \in C$ and $\omega = (\omega_1, \omega_2, \dots, \omega_t)^T$ be the weight vector of attributes, where $\omega_j \in [0, 1]$, $j = 1, 2, \dots, t$, and $\sum_{j=1}^t \omega_j = 1$.

Step 2: Eq(36) is used to calculate the fuzzy complementary judgment matrix $R_{C_j}(\alpha), C_j \in C$.

Step 3: Utilize the Eq(42) calculate the weighted fuzzy complementary judgment matrix $\bar{R}(\alpha)$.

Step 4: Eq(5) is used to calculate the weight $\beta_i, (i = 1, 2, 3, 4)$ of each alternatives.

Step 5: Rank all the alternatives $A_i, (i = 1, 2, \dots, s)$ Select the best alternative(s).

Step 6: End.

V. ILLUSTRATIVE EXAMPLE

In this section, a numerical example is given to show the feasibility and validity of the proposed the WA_M operator. We analyze an investment selection multiple attribute decision making problem.

Example [43]. A board of directors of an enterprise consists of five numbers, and they are planning to invest a sum of money in a suitable project of huge potential over the next 5 years. Suppose four possible projects $A = (A_1, A_2, A_3, A_4)$ will be evaluated by the board of directors. To evaluate and rank these projects, four criteria are suggested by the Balanced Score card methodology (it should be noted that all of them are of the maximization type), that is, C_1 : financial perspective, C_2 :

the customer satisfaction, C_3 : internal business process perspective, and C_4 : learning and growth perspective. The weight vector of the criteria is given as

$\omega = (0.2, 0.3, 0.15, 0.35)^T$. Assessment values given by the decision makers are displayed in the hesitant fuzzy decision matrix in Table 1.

TABLE I. HESITANT FUZZY DECISION MATRIX

	C_1	C_2	C_3	C_4
A_1	{0.2, 0.4, 0.7}	{0.2, 0.6, 0.8}	{0.2, 0.3, 0.6, 0.7, 0.9}	{0.3, 0.4, 0.5, 0.7, 0.8}
A_2	{0.2, 0.4, 0.7, 0.9}	{0.1, 0.2, 0.4, 0.5}	{0.3, 0.4, 0.6, 0.9}	{0.5, 0.6, 0.8, 0.9}
A_3	{0.3, 0.5, 0.6, 0.7}	{0.2, 0.4, 0.5, 0.6}	{0.3, 0.5, 0.7, 0.8}	{0.2, 0.5, 0.6, 0.7}
A_4	{0.3, 0.5, 0.6}	{0.2, 0.4}	{0.5, 0.6, 0.7}	{0.8, 0.9}

Step 1: Construct a hesitant fuzzy decision-matrix $H = (h_{ij})_{s \times t}$ according to the decision information given by the experts as Table 1.

Step 2: Eq(36) is used to calculate the fuzzy complementary judgment matrix $R_{C_j}(\alpha), C_j \in C$ as follows:

$$R_{C_1}(1) = \begin{bmatrix} 0.5 & 0 & 0.1075 & 0.3573 \\ 1 & 0.5 & 0.6071 & 0.8569 \\ 0.8925 & 0.3929 & 0.5 & 0.7498 \\ 0.6427 & 0.1431 & 0.2502 & 0.5 \end{bmatrix} \quad (46)$$

$$R_{C_2}(1) = \begin{bmatrix} 0.5 & 0.9998 & 0.7320 & 1 \\ 0.0002 & 0.5 & 0.2322 & 0.5 \\ 0.2680 & 0.7678 & 0.5 & 0.7678 \\ 0 & 0.5 & 0.2322 & 0.5 \end{bmatrix} \quad (47)$$

$$R_{C_3}(1) = \begin{bmatrix} 0.5 & 0.4167 & 0.2083 & 0 \\ 0.5833 & 0.5 & 0.2917 & 0.0833 \\ 0.7917 & 0.7083 & 0.5 & 0.2917 \\ 1 & 0.9167 & 0.7083 & 0.5 \end{bmatrix} \quad (48)$$

$$R_{C_4}(1) = \begin{bmatrix} 0.5 & 0.2714 & 0.5571 & 0.0571 \\ 0.7286 & 0.5 & 0.7857 & 0.2857 \\ 0.4429 & 0.2143 & 0.5 & 0 \\ 0.9429 & 0.7143 & 1 & 0.5 \end{bmatrix} \quad (49)$$

Step 3: Utilize the Eq(42) calculate the weighted fuzzy complementary judgment matrix $\bar{R}(\alpha)$ as follows:

$$\bar{R}(1) = \begin{bmatrix} 0.5 & 0.4574 & 0.4673 & 0.3914 \\ 0.5426 & 0.5 & 0.5098 & 0.4339 \\ 0.5327 & 0.4902 & 0.5 & 0.4241 \\ 0.6086 & 0.5661 & 0.5759 & 0.5 \end{bmatrix} \quad (50)$$

Step 4: Eq(5) is used to calculate the weight $\beta_i, (i = 1, 2, 3, 4)$ of each alternatives as follows:

$$\beta_1 = 0.2270, \quad \beta_2 = 0.2483, \quad \beta_3 = 0.2434, \\ \beta_4 = 0.2813.$$

Step 5: Rank all the alternatives $A_i, (i = 1, 2, \dots, s)$, Select the best alternative(s):

$$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1.$$

Step 6: End.

VI. CONCLUSIONS

In this paper, we constructed fuzzy complementary judgment matrix which has additive consistency and then we generalized it to the general model. Based on the presented fuzzy complementary judgment matrix and hesitant fuzzy average operator, a new multiple attribute decision making approach was obtained. Finally, an illustrative example was given to show the rationality of the proposed method.

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