

Hamilton Cycles in Intersection Graphs of Bases of Matroids

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Abstract—The intersection graph of bases of a matroid $M=(E, B)$ is a graph G with vertex set $V(G)$ and edge set $E(G)$ such that $V(G)=B$ and $E(G)=BB'$, where the same notation is used for the vertices of G and the bases of M , B and B' has no intersection. In this paper, we prove that for any given edge e of G , the intersection graph G of bases of a matroid M with rank at least 2 has a Hamilton cycle containing edge e and another Hamilton cycle avoiding edge e .

Keywords—matroid; intersection graph; Hamilton cycle

I. INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A matroid $M = (E, B)$ is a finite set E together with a nonempty collection $B(M)$ of subsets of E that satisfies the following condition: for any $B, B' \in B(M)$ with $|B|=|B'|$ and for any $e \in B \setminus B'$, there exists $e' \in B' \setminus B$ such that $(B \setminus e) \cup e' \in B(M)$. Each member of $B(M)$ is called a base of M . An element contained in every base is called a coloop, and an element contained in no base is called a loop. A matroid without loops and 2-circuits is called a simple matroid. The rank r of a matroid is the number of elements in a base. We denote the uniform matroid of rank m on an n -element set by $U_{m,n}$.

The base graph of a matroid $M=(E, B)$ is the graph $G'=G'(M)$ with vertex set $V(G'(M))=B(M)$ and edge set $E(G'(M))=\{BB': B, B' \in B(M) \text{ and } |B \setminus B'|=1\}$, where the same notation is used for the vertices of $G'(M)$ and the bases of M . The basic properties and characterizations of base graphs of matroids can be found in [1] and [2].

Liu and Li [3-7] studied the properties of circuit graphs of matroids. The circuit graph of a matroid M is a graph $G''=G''(M)$ with vertex set $V(G''(M))=C(M)$ and edge set $E(G'')=\{CC': |C \setminus C'|=1, C, C' \in C(M)\}$, where the same notation is used for the vertices of $G''(M)$ and the circuits of M . Next, we extend base graphs into a family of denser graphs by relaxing the requirement of vertices adjacency as follows: the *intersection graph* for bases of a matroid $M=(E, B)$ is the graph, denoted by $G^I(M)$ with vertex set $V(G^I(M))=B(M)$ and edge set $E(G^I(M))=\{BB': |B \cap B'|=1, B, B' \in B(M)\}$.

For $r(M)=1$, the intersection of any two bases of M is empty and thus the intersection graph $G^I(M)$ with rank $r(M)=1$ is a collection of $|B|$ isolated vertices. Clearly, for $r(M) \geq 2$, the intersection graph $G^I(M)$ contains the base graph $G'(M)$ as a connected spanning subgraph. In particular, for $r(M)=2$, the

intersection graph $G^I(M)$ is exactly the base graph G' of matroid M . The intersection graph $G^I(M)$ for bases of matroid $U_{2,4}$ is shown in Fig.1.

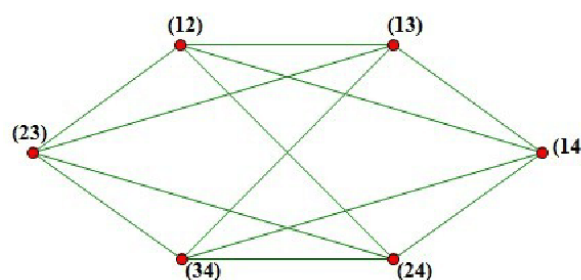


FIGURE 1. THE INTERSECTION GRAPH OF MATROID $U_{2,4}$

The problem of Hamilton cycles in base graphs of matroids have been investigated by many researchers. Cummins [8] showed that the base graph of a matroid with at least three vertices has a Hamilton cycle. Bondy [9] showed not only that every base graph is Hamiltonian, but also that most are pancyclic. Holzmman and Harary [10] showed that for every edge in the base graph of a matroid there is a Hamilton cycle containing it and another Hamilton cycle avoiding it. In [11], we give the definition of intersection graph for bases of a matroid $M=(E, B)$ and showed that the intersection graph $G^I(M)$ for bases of a simple matroid M with rank $r(M) \geq 2$ has at least two edge-disjoint Hamilton cycles whenever $|V(G^I(M))| \geq 5$.

Terminology and notations not defined here can be found in [12-13].

II. THE MAIN RESULTS

In this paper, we prove that for any given $e \in E(G^I(M))$, the intersection graph $G^I(M)$ of bases of a matroid M with rank $r(M) \geq 2$ has a Hamilton cycle containing e and another Hamilton cycle avoiding e whenever $|V(G^I(M))| \geq 4$. To prove the main theorems, we need the following lemmas.

Lemma 1 [13] If B_1 and B_2 are bases of a matroid M and $e \in B_1 \setminus B_2$, then there exists an element $f \in B_2 \setminus B_1$ such that both $(B_1 \setminus \{e\}) \cup \{f\}$ and $(B_2 \setminus \{f\}) \cup \{e\}$ are bases of M .

A graph G is positively Hamiltonian, written $G \in H^+$, if for every edge of G , there is a Hamilton circuit containing it. G is negatively Hamiltonian, written $G \in H^-$, if for every edge of G , there is a Hamilton circuit avoiding it. When $G \in H^+$ and $G \in H^-$, we say that G is *uniformly Hamiltonian*.

Lemma 2 [10] Let $M=(E, B)$ be a matroid on E and $G'=G'(M)$ be the matroid base graph of M . If $|V(G')| \geq 3$, then G' is positively Hamiltonian.

Lemma 3 [10] Let $M=(E, B)$ be a matroid on E and $G'=G'(M)$ be the matroid base graph of M . If $|V(G')| \geq 4$, then G' is negatively Hamiltonian.

In the following, we assume that matroid M satisfies that $r(M) \geq 2$ and $|B| \geq 3$. For any $e \in E \setminus B$, $B \in \{e\}$ contains a unique basic circuit, denoted by $C(e, B)$, and we use B_e and \bar{B}_e to denote the bases containing e and avoiding e , respectively.

Lemma 4 Let $M=(E, B)$ be a matroid with $|E|=n$ and $r(M)=r$. If M has no coloops, then $|\bar{B}_e| \geq n-r$ for any $e \in E$.

Proof Since M does not have coloops, there exists a base B of M such that $e \in B$. For any element $f \in (E \setminus B) \setminus \{e\}$, let $C(f, B) \in B \setminus \{f\}$ be a basic circuit of M with respect to the base B . Then exists an element $f' \in C(f, B) \setminus \{f\}$ such that $(B \setminus \{f\}) \cup \{f'\} \in B(M)$ and $e \notin (B \cup \{f\}) \setminus \{f'\}$. So there are at least $|(E \setminus B) \setminus \{e\}| + 1 = n-r$ bases of M avoiding e .

Let $e \in E$ and let G_1^e and G_2^e be subgraphs of $G^I(M)$ induced by \bar{B}_e and B_e , respectively. By the definition of intersection graphs, we have the following lemma immediately.

Lemma 5 Let $M=(E, B)$ be a matroid on E and $G^I(M)$ be the intersection graph of bases of M . For $e \in E$, G_1^e is the intersection graphs of bases of $M \setminus e$ and $G_2^e = G^I(M) - V(G_1^e)$.

It is easy to see that G_2^e is a complete graph induced by the vertices containing e and $|V(G_2^e)| = |B_e|$. The notations G_1^e and G_2^e keep the same meaning throughout the paper for a given $e \in E$.

Lemma 6 Let $M=(E, B)$ be a matroid on E without coloops and with rank $r(M) \geq 3$. Let $G^I(M)$ be the intersection graph of bases of M . For any edge $B_1 B_2 \in E(G^I(M))$, there exists an induced subgraph H with vertex set $V(H) = \{B_1, B_2, B_3, B_4\}$

such that $B_1 B_2 \in E(H)$ and $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$ for some $e \in E$. Furthermore, H is isomorphic to K_4 .

Proof Note that $|E| \geq 4$ by the hypothesis. Obviously, if $|E| = n < 2r$, then $G^I(M) \cong K_m$ ($m \geq 4$) and thus the lemma holds. So we only need to consider the case that $|E| = n \geq 2r \geq 6$. By the definition of intersection graph of bases of matroid and $B_1 B_2 \in E(G^I(M))$, we have $1 \leq |B_1 \cap B_2| \leq r-1$. Suppose that $|B_1 \cap B_2| = r-1 \geq 2$. Then $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$. Set $B_1 = A \cup \{x\}$ and $B_2 = A \cup \{y\}$ (here $|A| \geq 2$). Since $r^*(M) = |E| - r(M) \geq 2r - r \geq r$, there exists an element $b \in E \setminus (B_1 \cup B_2)$ such that $C_1 = C(b, B_1) \subseteq B_1 \cup \{b\}$ and $C_2 = C(b, B_2) \subseteq B_2 \cup \{b\}$. For every element $b \in E \setminus (B_1 \cup B_2)$, if $(C_1 \cup C_2) \cap A = \emptyset$, then either every element of A is a coloop or $A = \emptyset$, which contradicts to the hypothesis of the lemma. So there exists an element $a \in A$ such that $a \in (C_1 \cup C_2) \cap A$. We consider the following three cases.

Case 1 $a \in (C_1 \cap C_2)$.

It is obvious that $B_4 = (B_1 \cup \{b\}) \setminus \{a\} = (A \setminus \{a\}) \cup \{x, b\}$ and $B_3 = (B_2 \cup \{b\}) \setminus \{a\} = (A \setminus \{a\}) \cup \{y, b\}$ are bases of M . Thus there exists an element $e = x \in E$ such that $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$.

Case 2 $a \in C_2 \setminus C_1$.

Clearly, $B_4 = (B_1 \cup \{b\}) \setminus \{a\} = (A \setminus \{a\}) \cup \{x, b\}$ is a base of M . Next we show that $y \notin C_2$ by contradiction. Suppose that $y \in C_2$. Then $C_2 \subseteq (B_1 \cup B_2) \cup \{b\} \subseteq B_1 \cup \{b\}$. But $C_1 \subseteq B_1 \cup \{b\}$ and there is only one circuit in $B_1 \cup \{b\}$. So we have $C_1 = C_2$, which is a contradiction. Set $B_3 = (B_2 \cup \{b\}) \setminus \{y\} = A \cup \{b\}$. Then there exists an element $e = x \in E$ such that $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$.

Case 3 $a \in C_1 \setminus C_2$.

For this case, the arguments are similar to that of Case 2, and we can show that $x \in C_1$.

Thus obtain bases $B_4 = (B_1 \cup \{b\}) \setminus \{x\} = A \cup \{b\}$ and $B_3 = (B_2 \cup \{b\}) \setminus \{a\} = (A \setminus \{a\}) \cup \{x, b\}$. Then there exists an element $e = y \in E$ such that $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$. Without loss of generality, we may assume $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$ (if necessary, we can interchange B_1 and B_2 , B_3 and B_4).

Furthermore, it is easy to check that $(B_1 \cap B_2) \setminus \{a\} = A \setminus \{a\} \subset B_i$ ($i=\{1, 2, 3, 4\}$), so the subgraph of $G^I(M)$ induced by $\{B_1, B_2, B_3, B_4\}$ is isomorphic to K_4 by the definition of intersection graph $G^I(M)$ of bases of matroid M .

Suppose that $1 \leq |B_1 \cap B_2| \leq r-1$. Then there exists an element $e \in B_1 \setminus B_2$. By Lemma 1, there also exists an element $f \in B_2 \setminus B_1$ such that $B_3 = (B_1 \setminus \{e\}) \cup \{f\}$ and $B_4 = (B_2 \setminus \{f\}) \cup \{e\}$ are bases of matroid M . Then $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$.

It is not hard to see that the subgraph induced by $\{B_1, B_2, B_3, B_4\}$ is also isomorphic to K_4 because $(B_1 \cap B_2) \subset B_i$ ($i=\{1, 2, 3, 4\}$). Hence, for any edge $B_1B_2 \in E(G^I(M))$, we can find an induced subgraph H with vertex set $V(H) = \{B_1, B_2, B_3, B_4\}$ such that $B_1B_2 \in E(H)$ and $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$ for some $e \in E$. Furthermore, H is isomorphic to K_4 .

Now we proceed to the main theorems.

Theorem 1 Let $M=(E, B)$ be a matroid without coloops on E and $G=G^I(M)$ be the intersection graph of bases of M . If $|V(G^I(M))| \geq 3$, then $G^I(M)$ is positively Hamiltonian.

Proof We prove the theorem by induction on $|E|=n$. Note that we have $|E| \geq 3$ and $r(M) \geq 2$. If $|E|=3$ and $r(M)=2$, then $M \cong U_{2,3}$ and it is easy to see that $G=G^I(M) \cong K_3$, so the theorem holds. Suppose that the result is holds for $|E| \leq n-1$. We prove that the result also holds for $|E|=n \geq 3$. Obviously, if $|E|=n < 2r$, then $G=G^I(M) \cong K_m$ ($m \geq 3$) and thus the theorem holds.

In fact, if $r(M)=2$, then the intersection graph $G=G^I(M)$ is isomorphic to the matroid base graph of M . The theorem holds by Lemma 2. So we only need to consider the case that $r(M) \geq 3$ and $|E|=n \geq 2r \geq 6$. Let B_1B_2 be any edge of $G^I(M)$.

By Lemma 6, for any edge $B_1B_2 \in E(G^I(M))$, we can find an induced subgraph H with vertex set $V(H) = \{B_1, B_2, B_3, B_4\}$ such that $B_1B_2 \in E(H)$ and $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$ for some $e \in E$. Furthermore, H is isomorphic to K_4 . By Lemma 4, we have $|V(G_1^e)| = |\overline{B_e}| \geq n-r \geq r \geq 3$. If $M \setminus e$ has coloops, then G_1^e is a complete graph K_m with order $m \geq 3$. So G_1^e has a Hamilton cycle C_1 containing B_2B_3 . If $M \setminus e$ does not have coloops, then the induction hypothesis assures the existence of a Hamilton cycle C_1 of G_1^e containing B_2B_3 . It is easy to see that $G_2^e =$

$G^I(M) - V(G_1^e)$ is a complete graph and thus there is a Hamilton path P in G_2^e connecting B_1 and B_4 . Thus $C = (C_1 - B_2B_3) \cup B_3B_4 \cup P \cup B_1B_2$ is a Hamilton cycle which containing $B_1B_2 \in E(G^I(M))$ (see Fig. 2). The proof is complete.

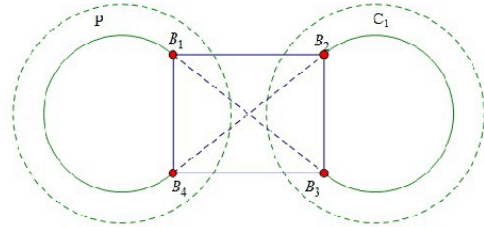


FIGURE II. HAMILTON CYCLE CONTAINING B_1B_2

Theorem 2 Let $M=(E, B)$ be a matroid without coloops on E and $G=G^I(M)$ be the intersection graph of bases of M . If $|V(G^I(M))| \geq 4$, then $G^I(M)$ is negatively Hamiltonian.

Proof If $r(M)=2$, then the intersection graph $G^I(M)$ of bases of M is isomorphic to the base graph $G^I(M)$ of M and thus the theorem holds by Lemma 3. Next we assume that $r(M) \geq 3$. Suppose that $|E|=4$ or $|E|=5$ and $r(M)=3$. Then the intersection graph $G^I(M)$ is a complete graph K_m with order $m \geq 4$, the theorem clearly holds. Moreover, if $|E|=n < 2r$, then $G^I(M) \cong K_m$ ($m \geq 3$) and so the theorem also holds. Now we consider the remaining case that $r(M) \geq 3$ and $|E|=n \geq 2r \geq 6$. Let B_1B_2 be any edge of $G^I(M)$. By Lemma 6, for any edge $B_1B_2 \in E(G^I(M))$, we can find an induced subgraph H with vertex set $V(H) = \{B_1, B_2, B_3, B_4\}$ such that $B_1B_2 \in E(H)$ and $\{B_2, B_3\} \subseteq V(G_1^e)$, $\{B_1, B_4\} \subseteq V(G_2^e)$ for some $e \in E$. Furthermore, H is isomorphic to K_4 . In fact, the induced subgraph H is isomorphic to K_4 implies that $B_1B_3 \in E(G^I(M))$ and $B_2B_4 \in E(G^I(M))$.

By Lemma 4, we have $|V(G_1^e)| = |\overline{B_e}| \geq n-r \geq r \geq 3$. If $M \setminus e$ has coloops, then G_1^e is a complete graph K_m with order $m \geq 3$. So G_1^e has a Hamilton cycle C_1 containing B_2B_3 . If $M \setminus e$ does not have coloops, then by Theorem 1, there exists a Hamilton cycle C_1 of G_1^e containing B_2B_3 .

Obviously, $G_2^e = G^I(M) - V(G_1^e)$ is a complete graph and there is a Hamilton path P in G_2^e connecting B_1 and B_4 . Thus $C' = (C_1 - B_2B_3) \cup B_1B_3 \cup P \cup B_4B_2$ is a Hamilton cycle avoiding $B_1B_2 \in E(G^I(M))$ (see Fig. 3).

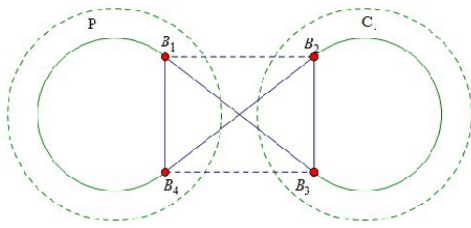


FIGURE III. HAMILTON CYCLE AVOIDING B_1B_2

Corollary 1 Let $M=(E, B)$ be a matroid on E and $G=G^I(M)$ be the intersection graph of bases of M . If $|V(G^I(M))| \geq 4$, then $G^I(M)$ is uniformly Hamiltonian.

Remark Note that the results in this paper are best possible in the condition $|V(G^I(M))| \geq 4$ in the theorem is necessary. In fact, if $|V(G^I(M))| < 3$, then $G^I(M)$ has no cycles; if $|V(G^I(M))| = 3$, then $G^I(M)$ is isomorphic to 3-cycle which is not negatively Hamiltonian.

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