Modulus-based Matrix Multi-splitting Multi-Parameter Methods for a Class of Weakly Nonlinear Complementarity Problem

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Abstract—The modulus-based matrix multi-splitting multi-parameter methods for solving a class of weakly nonlinear complementarity problem are presented. The weaker convergence conditions are analyzed. The numerical results show that the new methods are efficient.

Keywords—weakly nonlinear complementarity problem; modulus-based matrix multi-splitting; multi-parameter

I. INTRODUCTION

In this paper, we consider the following weakly nonlinear complementarity problem: for \( N \in \mathbb{R}^{n \times n} \), \( q \in \mathbb{R}^n \), finding a pair of vectors \( u \), \( w \in \mathbb{R}^n \) such that

\[
w = Nu + q + m(u) \geq 0, \quad w \geq 0, \quad u^T (Nu + q + m(u)) = 0,
\]

where the nonlinear term \( m(u) = (m_1(u_1), m_2(u_2), \ldots, m_n(u_n))^T \), and \( m_i(u_i) \) is a real function of a domain in real numbers, and satisfies that \( \frac{\partial m_i}{\partial u_i} \geq 0 \).

The complementarity problem has been widely used in the research of operation research, optimal control, engineering problems, economics and mechanics, etc. A series of methods is proposed to solve the linear complementarity problem; see [2,3,4,5,9]. By splitting the coefficient matrix, the modulus-based matrix splitting iterative methods are proposed see [6,7,8]. Hong [8] and Li [11] proposed a class of modulus-based iterative methods for solving the implicit complementarity problems. Zhang [13] summarized the modulus-based matrix multi-splitting methods for solving a class of nonlinear complementarity problems, and obtained the weaker convergence conditions of the new method. Recently, Zhang [15] further studied modulus-based synchronous block multi-splitting multi-parameter methods for linear complementarity problems, and provide the wider convergence results and the optimal relaxation parameters.

In this paper, we present the modulus-based multi-splitting multi-parameter accelerated overrelaxation iterative methods (MSMMAOR) to solving the weakly nonlinear complementarity problem (1), and give the weaker convergence conditions.

The rest of the paper is as follows. We give some necessary notations and lemmas in Section 2, and propose the modulus-based synchronous multi-splitting multi-parameter accelerated overrelaxation iterative methods for solving the weakly nonlinear complementarity in Section 3. The convergence of the new methods is proved in Section 4. The numerical results are shown in Section 5, and some concluding remarks are given in Section 6.

II. PRELIMINARIES

Given the matrix \( N = (n_{ij}) \in \mathbb{R}^{n \times n} \), its comparison matrix \( \langle N \rangle = (\langle n_{ij} \rangle) \in \mathbb{R}^{n \times n} \) is defined by

\[
\langle n_{ij} \rangle = \begin{cases} |p_{ij}|, & i = j, \\ -|p_{ij}|, & i \neq j. \end{cases}
\]

A matrix \( N \) is called a \( Z \)-matrix if all of its off-diagonal entries are nonpositive. The set of all \( Z \)-matrix is denoted as \( Z^{\text{non}} \). A nonsingular matrix called an \( M \)-matrix if \( N \) is a \( Z \)-matrix and \( N^{-1} \geq 0 \). A matrix is an \( H \)-matrix if its comparison matrix \( \langle N \rangle \) is an \( M \)-matrix, and it is an \( H^+ \)-matrix if it is an \( H \)-matrix with positive diagonal entries. Let \( N \), \( L \in \mathbb{R}^{n \times n} \) be \( M \)-matrices, \( \Omega \in \mathbb{R}^{n \times n} \) be a positive matrix, and \( C \in \mathbb{R}^{n \times n} \). Then, \( N \preceq L \) implies \( L^{-1} \preceq N^{-1} \), and \( N \preceq C \preceq \Omega \) implies that \( C \) is an \( M \)-matrix. Let \( N = Q - W \) is called a splitting of the matrix \( N \) if \( Q \) is nonsingular. The...
splitting $N = Q - W$ is called an $H$- compatible splitting if
\[ \langle N \rangle = \langle Q \rangle - |W|. \]

Lemma 2.1 ([1])

Let $N \in \mathbb{Z}^{+}^{n}$ which has all positive diagonal entries. $N$ is an $M$- matrix if and only if $\rho(D^{-1}B) < 1$, where $D = \text{diag}(N), B = D - N$.

Lemma 2.2 ([1])

Let $N \in \mathbb{R}^{+}^{n \times n}$ be an $H$- matrix, $D = \text{diag}(N)$ be the diagonal matrix of $N$, and $B = D - N$. Then, the following statements hold true:

1. $N$ is nonsingular;
2. $N^{-1} \leq \langle N \rangle^{-1}$;
3. $|D|$ is nonsingular and $\rho(D^{-1}B) < 1$.

Lemma 2.3 (Perron-Frobenius Theorem)

Let $N \in \mathbb{R}^{+}^{n \times n}$ be a irreducible nonnegative matrix, then

1. The spectral radius $\rho(N)$ is a positive real eigenvalue of matrix $N$;
2. The eigenvector $x$ of $\rho(N)$ is a positive vector;
3. $\rho(N)$ is a simple eigenvalue of $N$;
4. If any element of matrix $N$ is increased, $\rho(N)$ increases.

Lemma 2.4 ([12])

Let $N = Q - W$ be a splitting of the matrix $N \in \mathbb{R}^{+}^{n \times n}$, and $\Omega$ be a positive diagonal matrix. Then, for the nonlinear complementarity problem (1), the following statements hold true:

(a) If $(u, w)$ is a solution of the nonlinear complementarity problem (1), then $x = \frac{\gamma}{2}(u - \Omega^{-1} \cdot w)$ satisfies the implicit fixed-point equation
\[ (\Omega + Q)x = Wx + (\Omega - N)|x| - \gamma \left[ q + m \left( \frac{1}{\gamma} |x| + x \right) \right]. \tag{2} \]

(b) If $x$ satisfies the implicit fixed-point equation (2), then
\[ u = \frac{1}{\gamma}(|x| + x), \quad w = \frac{1}{\gamma} \Omega(|x| - x). \]

are solutions of the weakly nonlinear complementarity problem (1).

III. MODULUS-BASED MULTI-SPLITTING MULTIPARAMETER ACCELERATED OVER-RELAXATION ITERATIVE METHOD

Let $\ell (\ell \leq n)$ be a given positive integer and $N \in \mathbb{R}^{+}^{n \times n}$. Let $L_k$ be a strictly lower-triangular matrix and $U_k = D - L_k$ for $k = 1, 2, \ldots, \ell$, where $D = \text{diag}(N)$. Note that $U_k$ has zero diagonal but needs not be upper triangular. Let the weighting matrices $E_k \in \mathbb{R}^{+}^{n \times n}$ ($n = 1, \ldots, \ell$) be nonnegative diagonal matrices satisfying $\sum_{k=1}^{\ell} E_k = I$ (the identity matrix). Then, the collection of triples $(D - L_k, U_k, E_k)$ ($k = 1, 2, \ldots, \ell$) is called a triangular multi-splitting of the matrix $N$.

Let $N = Q_k - W_k$ be a multi-splitting of $N \in \mathbb{R}^{+}^{n \times n}$,
\[ \left\{ \begin{align*}
Q_k &= \frac{1}{\alpha}(D - \beta L_k) \\
W_k &= \frac{1}{\alpha} \left[ (1 - \alpha)D + (\alpha - \beta)L_k + \alpha U_k \right]
\end{align*} \right. \]
for $k = 1, 2, \ldots, \ell$, then from the system of equations (2), we get
\[ (\alpha \Omega + D - \beta L_k)x = \left[ (1 - \alpha)D + (\alpha - \beta)L_k + \alpha U_k \right]x + \alpha(\Omega - N)|x| - \alpha \gamma \left[ q + m \left( \frac{1}{\gamma} |x| + x \right) \right]. \tag{3} \]

The following is the MSMAOR method for solving the weakly nonlinear complementarity problem (1).

Method 3.1 (MSMAOR method)

Step 1: Given $x^{(0)} \in \mathbb{R}^n$, and $\varepsilon > 0$, set $m := 0$;

Step 2: On $\ell$ different processors corresponding fixed point system, respectively.
\[ (\alpha \Omega + D - \beta L_k)x^{(m+1,k)} = \left[ (1 - \alpha)D + (\alpha - \beta)L_k + \alpha U_k \right]x^{(m)} + \alpha(\Omega - N)x^{(m)} - \alpha \gamma \left[ q + m \left( \frac{1}{\gamma} |x^{(m)}| + x^{(m)} \right) \right]; \]

Step 3: Let
\[ x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m+1,k)}, \quad \text{and} \quad u^{(m+1)} = \frac{1}{\gamma} \left( |x^{(m+1)}| + x^{(m+1)} \right). \]
If \( RES\left(u^{(m+1)}\right) < \varepsilon \), then stop; otherwise, set \( m := m + 1 \), and return to Step 2.

Let \( \alpha = \alpha_k \), \( \beta = \beta_k \), the following is MSMMAOR method for solving weakly nonlinear complementarity problem.

**Method 3.2(MSMMAOR method)**

Step 1: Given \( x^{(0)} \in \mathbb{R}^n \), and \( \varepsilon > 0 \), set \( m := 0 \).

Step 2: On \( t \) different processors corresponding fixed point system, respectively.

\[
\left( \alpha_k \Omega + D - \beta_k L_k \right) x^{(m+1,k)} = \left[ 1 - \alpha_k \right] D + \left[ \alpha_k \right] \Omega - N - a_k \varepsilon \left( x^{(m,k)} \right) - \alpha_k \gamma \left[ q + m \left( 1/\gamma \right) \left( x^{(m)} + x^{(m)} \right) \right].
\]

Step 3: Let

\[
x^{(m+1)} = \sum_{k=1}^{t} E_k \left( x^{(m+1,k)} \right), \text{and } u^{(m+1)} = \frac{1}{\gamma} \left( x^{(m+1)} + x^{(m+1)} \right).
\]

If \( RES\left(u^{(m+1)}\right) < \varepsilon \), then stop; otherwise, set \( m := m + 1 \), and return to Step 2.

IV. **CONVERGENCE**

In this section, we prove the convergence of MSMAOR and MSMMAOR. We assume that \( \left\{ u^*, w^* \right\} \) is a solution of the nonlinear complementarity problem (1). By Lemma 2.4, we know that \( x^* = \frac{1}{2}\left( u^* - \Omega^{-1} w^* \right) \) satisfies the implicit fixed-point equation (2).

Let \( N \) be an \( H_t \)-matrix, with the notations \( D = \text{diag}(N) \) and \( B = D - N \), by Lemma 2.2, we see that \( \rho(\left[D^{-1}\right]B) < 1 \).

Let \( J = D^{-1}[B] \) and \( J_e = J + ee^T \), where \( e = (1, 1, \ldots, 1) \) \( \in \mathbb{R}^t \) and \( \varepsilon > 0 \). Then, \( \left\{ N \right\} = D(J - J) \), and by Lemma 2.3, there exists a positive vector such that \( J_e y_e = \rho_e y_e \).

**Theorem 4.1**

Let \( N \in \mathbb{R}^{t \times t} \) be an \( H_e \)-matrix, with \( D = \text{diag}(N) \) and \( B = D - N \). Let \( \left( D - L_k, U_k, E_k \right) \) \( k = 1, 2, \ldots, t \) be a triangular multi-splitting of the matrix \( N \), and \( N = D - L_k - U_k \) satisfies \( \left\{ N \right\} = D - [L_k] - [U_k] \). Assume that \( \Omega \geq D + M_e \) is a positive diagonal matrix, \( \gamma \) is a positive constant, \( m(u) \) satisfies \( \frac{\partial m}{\partial u_i} \geq 0 \), and

\[
M_e = \text{diag} \left( \frac{\partial m_1}{\partial u_1}, \frac{\partial m_2}{\partial u_2}, \ldots, \frac{\partial m_n}{\partial u_n} \right). \]

Then, the iteration sequence \( \left\{ u^{(m)} \right\} \in \mathbb{R}^t \) generated by the MSMAOR iteration method converges to the unique
solution $u^*$ of the problem (1) for any initial vector $x^{(0)} \in \mathbb{R}^n$, provided the relaxation parameters $\alpha_k$ and $\beta_k$ ($k = 1, 2, \ldots, \ell$) satisfy
\[
0 < \beta_k \leq \alpha_k \leq 1 \text{ or } 0 < \beta_k \leq \frac{1}{\rho(D^{-1}[B])}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}[B])}.
\]

V. NUMERICAL RESULTS

In this section, we use an example to test the numerical effectiveness of Method 1. Let $RES(\min(z, w)) \leq 10^{-3}$, $n \in Z^+$ be the order of the matrix $N$, $d$ denote the iteration steps, and $CPU$ denote the iteration time.

Example 5.1

Consider the following nonlinear complementarity problem (1). $I$ is a unit matrix. Let $N \in \mathbb{R}^{n \times n}$ be an $H$-matrix,
\[
N = \begin{pmatrix} B & -0.5I & O & \cdots & O & O \\ -1.5I & B & -0.5I & \cdots & O & O \\ O & -1.5I & B & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -0.5I \\ O & O & \cdots & \cdots & -1.5I & B \end{pmatrix}
\]

where
\[
B = \begin{pmatrix} 4 & -1 & O & \cdots & O & O \\ -1 & 4 & -1 & \cdots & O & O \\ O & -1 & 4 & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & 4 & -1 \\ O & O & \cdots & -1 & 4 \end{pmatrix},
\]

and $q = (1, -1, \ldots, 1, -1)^T$, $m(u) = (u^3, u^2, \ldots, u^3, u^2)^T$.

The numerical results of MSMAOR method are shown in TABLE I, and the numerical results of MSMMAOR method are shown in TABLE II.

TABLE I. THE NUMERICAL RESULTS OF MSMAOR METHOD FOR EXAMPLE 5.1

<table>
<thead>
<tr>
<th>MSMAOR $(\alpha, \beta)$</th>
<th>n=2500</th>
<th>n=3600</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.07, 1.01)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell$</td>
<td>IT</td>
<td>CPU</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>0.289</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>0.131</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>0.096</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>0.071</td>
</tr>
</tbody>
</table>

TABLE II. THE NUMERICAL RESULTS OF MSMMAOR METHOD FOR EXAMPLE 5.1

<table>
<thead>
<tr>
<th>MSMMAOR $(\alpha, \beta)$</th>
<th>n=2500</th>
<th>n=3600</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.9, 0.8)</td>
<td>IT</td>
<td>CPU</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>(0.9, 0.8)</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>(1, 0.9)</td>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

From Tables I and II, we observe that the convergence conditions of Theorem 4.2 are weaker than Theorem 4.1 and MSMMAOR is better than MSMAOR.

VI. CONCLUSION

In this paper, we extend MSMMAOR to solve a class of weakly nonlinear complementarity problems, analyze its convergence and present its weaker convergent conditions than MSMAOR. The numerical results confirm the theory results and show that MSMMAOR is better than MSMAOR.

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