On the Oscillation of Impulsive Partial Fractional Differential Equations

Zhuo Qu, Siying Zhu, Meng Su and Anping Liu*
School of Mathematics and Physics, China University of Geosciences, Wuhan, Hubei, 430074, China
*Corresponding author

Abstract—In this paper, we investigate the oscillation properties of a class of impulsive partial fractional differential equations with several delays. Some sufficient conditions for oscillation of the solutions are obtained by employing integral transformation technique and differential inequality method, and an example is given to illustrate the main result.

Keywords—oscillation; impulsive; delays; partial fractional differential equations

I. INTRODUCTION

It is well known that many important mathematical models are described by differential equations containing fractional order derivatives. The theories of fractional differential equations and their applications have been investigated extensively, we can find numerous applications in viscoelasticity, electrochemistry, control, porousmedia, electromagnetic, etc. see[1-6]. The oscillatory behavior of ordinary differential equations, partial differential equations and impulsive partial differential equations have been investigated by many authors in recent years, see[12-20].

However, to the best of our knowledge very little is known regarding the oscillatory behavior of impulsive partial fractional differential equations with several delays up to now. Recently, A. Raheem and Md. Maqbul studied the oscillation behavior of solutions of a class of fractional ordinary differential equations and fractional partial differential equations have been investigated by many authors in the past, see [7-11]. The oscillatory behavior of various classes of fractional ordinary differential equations and impulsive partial differential equations has been investigated by many authors in recent years, see[12-20].

The purpose of this paper is to study the oscillation properties of the solutions to a class of impulsive partial fractional differential equations with several delays

\[
D^{\alpha}_{+}u(t,x)+b(t)D^{\sigma}_{+}u(t,x)=a(t)h(u(t,x))\Delta u(t,x)-
\sum_{k=1}^{n} q_k(t,x)f_k(u(t-\delta_k,x))=g(t,x), t \not= t_k, (t,x) \in \Omega, t \geq 0,
\]

\[
D^{\alpha}_{+}u(t_k^-, x)-D^{\alpha}_{+}u(t_k^+, x) = \sigma(t_k,x)D^{\sigma}_{+}u(t_k^-, x), t = t_k, k=1,2,\ldots,
\]

with the boundary condition

\[
\frac{\partial u(t,x)}{\partial n} = w(t,x,u(t,x)), (t,x) \in \Omega, t \not= t_k.
\]

Where \( \alpha \in (0,1) \) is a constant; \( D^{\alpha}_{+} \) is the Riemann-Liouville fractional derivative of order \( \alpha \) of \( u(t,x) \) with respect to \( t \); \( \Omega \) is a boundary domain in \( R^n \) with a smooth boundary \( \partial \Omega \) and \( \Omega = \Omega \cup \partial \Omega \); \( \sigma > 0 \). \( \Delta \) is Laplacian operator, and \( n \) is the unit exterior normal vector to \( \partial \Omega \); \( b(t), a(t) \in PC[R, \Omega] \), \( PC \) denote the class of functions which are piecewise continuous in \( t \) with discontinuities of first kind only at \( t = t_k \); \( \delta_k > 0 \). The solution \( u(t,x) \) of the problem (1) - (2) and \( D^{\alpha}_{+}u(t,x) \) are piecewise continuous with discontinuities of first kind only at \( t = t_k \), and left continuous at \( t = t_k \), \( k=1,2,\ldots \).

Throughout this paper, we assume that the following conditions hold:

(H1): \( F : R \rightarrow R \) is a continuous function such that \( F(u)/u \geq K \), for all \( u \neq 0 \); and \( K \) is a positive constant.

(H2): \( q(t,x) \in PC[R \times \Omega, R] \), and \( q(t) = \min_{1 \leq i \leq n} q_i(t,x) \).

(H3): \( g(t,x) \in PC[R \times \Omega, R] \).

(H4): \( h(u) \in C(R,R) \); \( uh'(u) \geq 0 \); \( w(t,x,u) \) is a piecewise continuous function, such that \( wh(t,x,u)h'(u) \leq 0 \).

(H5): \( \sigma : R \times \Omega \rightarrow R \), such that \( \sigma(t_k,x) \leq \alpha_k \).

(H6): The given numbers \( 0 < t_1 < \cdots < t_k < \cdots \), are such that \( \lim_{k \to \infty} t_k = +\infty \).

(H7): At the moments of impulsive the following relation is satisfied

\[
D^{\alpha}_{+}u(t_k^+, x) = D^{\alpha}_{+}u(t_k^-, x).
\]

For the sake of convenience, in this paper, we denote
\[ U(t) = \int_{x_0} u(t,x)dx, \quad G(t) = \int_{x_0} g(t,x)dx. \]  

\subsection{Preliminaries}

\textbf{Definition 2.1}

A nonzero solution \((x(t), u(t,x))\) of the problem (1)-(2) is said to be nonoscillatory in the domain \(G\) if there exists a number \(t_0 \geq 0\) such that \(u(t,x)\) has a constant sign for \((t,x) \in [t_0, +\infty) \times \Omega\). Otherwise, it is said to be oscillatory.

\textbf{Definition 2.2}

The Riemann-Liouville fractional partial derivative of order \(\alpha > 0\) with respect to \(t\) of a function \((x(t), u(t,x))\) on the half-axis \(\mathbb{R}_+\) is defined by
\[ D^\alpha_{x,t}u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-v)^{-\alpha} u(v,x)dv, \quad t > 0, \] (4)
where \(\alpha \in (0,1)\); \(\Gamma\) is gamma function, and 
\[ \Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds. \]

\textbf{Definition 2.3}

The Riemann-Liouville fractional derivative of order \(\alpha > 0\) of a function \(x: \mathbb{R}_+ \rightarrow \mathbb{R}\) on the half-axis \(\mathbb{R}_+\) is defined by
\[ D^\alpha_{x}x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_0^t (t-v)^{\alpha-1} x(v)dv, \quad t > 0, \] (5)
where \(\alpha \in (0,1)\); \(\Gamma\) is gamma function, \(\lfloor \alpha \rfloor\) is the ceiling function of \(\alpha\).

\textbf{Lemma 2.4}\(^{[3]}\)

Let \(0 < \alpha < 1\), \(m \in \mathbb{N}\) and \(D = d/dx\). If the fractional derivatives \((D^m_x)^\alpha y(x)\) and \((D^m u - \alpha a) y(x)\) exist, then
\[ (D^m D^\alpha_x y)(x) = (D^{m+a} u)(x) \] (6)

\subsection{Main Results}

\textbf{Theorem 3.1}

If impulsive fractional differential inequality
\[ D^\alpha_{x,t}U(t) + b(t)D^\alpha_{x,t}U(t) \leq -G(t), \] (7)
has no eventually positive solutions and impulsive fractional differential inequality
\[ D^\alpha_{x,j}U(t) + b(t)D^\alpha_{x,j}U(t) \geq -G(t), \] (9)
\[ D^\alpha_{x,j}U(t) \geq (1+\alpha\epsilon)D^\alpha_{x,j}U(t), k = 1, 2, \ldots, \] (10)
has no eventually negative solutions, then every nonzero solution \(u(t,x)\) of the problem (1) and (2) is oscillatory in the domain \(G\).

\textbf{Proof.} Suppose to contrary that \(u(t,x)\) be a nonzero solution of the problem (1) and (2) which is nonoscillatory in the domain \(G\). Without loss of generality, we assume that \(u(t,x)\) is an eventually positive solution of problem (1) and (2) in the domain \(G\), then there exists a \(t_0 \geq 0\) such that \(u(t,x) > 0, u(t-\delta_j) > 0\), for \((t,x) \in [t_0, +\infty) \times \Omega\).

Case 1: \(t \neq t_k\). Integrating the first equation of problem (1) with respect to \(x\) over the domain \(\Omega\), we have
\[ \int_{x_0} \int_{t_0}^t D^\alpha_{x,t}u(t,x)dx + b(t)\int_{x_0}^t D^\alpha_{x,t}U(t)dx = a(t)\int_{x_0} h(u(t,x))dx - \int_{x_0} \sum_{i=1}^m q_i(t,x)f_i(u(t-\delta_j, x))dx \] (11)
\[ - \int_{x_0} g(t,x)dx \]

By using Green's formula, combing boundary condition (2) and assumption (H4), we obtain
\[ \int_{x_0} h(u)\Delta u(t,x)dx = \int_{x_0} \nabla h(u) \cdot \nabla u(t,x)dx - \int_{x_0} h'(u)[\nabla u]^2 dx \] (12)
\[ = \int_{x_0} h(u)\Delta u(t,x, u)dx - \int_{x_0} h'(u)[\nabla u]^2 dx \leq 0 \]

According to assumption (H1) and (H2), we have
\[ \int_{x_0} \sum_{i=1}^m q_i(t,x)f_i(u(t-\delta_j, x))dx \] (13)
\[ \geq \int_{x_0} \sum_{i=1}^m q_i(t,x)u(t-\delta_j, x)dx \geq 0, t \geq t_0. \]

And according to Lemma 2.4, combing (11)-(13), we can easily obtain...
\[ D^\alpha_{x}U(t)+b(t)D^\alpha_{x}U(t) \leq \sum_{i=1}^{n} k_i g_i(t) \int_{\Omega} u(t-\delta,x)dx \]
\[ \leq \sum_{i=1}^{n} k_i g_i(t) \int_{\Omega} u(t-\delta,x)dx - G(t) \leq -G(t), t \geq t_0. \]

Case 2: \( t = t_i \). Integrating the second equation of problem (1) with respect to \( x \) over the domain \( \Omega \), and according to assumption (H5), we have

\[ D^\alpha_{x}U(t_i^+) = D^\alpha_{x}U(t_i^-) \int_{\Omega} u(t_i^-+x)dx \]
\[ \leq (1+\alpha) D^\alpha_{x}U(t_i^-+1) = (1+\alpha) D^\alpha_{x}U(t_i^+) \]

Thus impulsive fractional differential inequality (14) and (15) imply that the function \( U(t) = \int_{\Omega} u(t,x)dx \) is an eventually positive solution of the fractional impulsive differential inequality (7) and (8) which contradicts the conditions of the theorem.

On the other hand, if \( u(t,x) \) is an eventually negative solution of the problem (1) and (2) in the domain \( G \), then each nonzero solution of problem (1) and (2) is oscillatory.

D. Theorem 3.4

If for some \( \tau_2 > 0 \),

\[ \int_{t_2}^{\infty} \exp(-b(\sigma)d\sigma)d\sigma = \infty, \]

further, for some \( \tau_1 \geq 0 \),

\[ \limsup_{t \to \infty} \prod_{t_i \in \tau_1} (1+\alpha) \exp\left(-\int_{t_i}^{t} b(\sigma)d\sigma\right)G(s)ds = \infty, \]

and

\[ \liminf_{t \to \infty} \prod_{t_i \in \tau_1} (1+\alpha) \exp(-\int_{t_i}^{t} b(\sigma)d\sigma)G(s)ds = -\infty, \]

then each nonzero solution of problem (1) and (2) is oscillatory in the domain \( G \).

Proof. To prove the theorem, it is sufficient to prove that the impulsive fractional differential inequality (7) and (8) has no eventually positive solutions, and the impulsive fractional differential inequality (9) and (10) has no eventually negative solutions. Suppose to contrary that the impulsive fractional differential inequality (7) and (8) has an eventually positive solution \( U(t) \), then there exists \( \tau_1 \geq 0 \) such that \( U(t) > 0 \), \( U(t) - \delta_0 > 0, G(t) > 0, t \geq \tau_1 \).

Let

\[ v(t,x) = \exp(\int_{t_0}^{t} b(s)ds). \]

By using Lemma 2.4 and the impulsive fractional differential inequality (7), we have

\[ [(D^\alpha_{x}U)v(t)] = (D^\alpha_{x}U)(v(t)+b(t)(D^\alpha_{x}U)v(t) \leq -G(t)v(t) < 0. \]

Thus \( (D^\alpha_{x}U)v(t) \) is strictly decreasing for \( t \geq \tau_1 \) and is eventually of constant sign. Since \( v(t) > 0 \), for \( t \geq \tau_1 \), we know that \( (D^\alpha_{x}U)(t) \) is eventually of constant sign. Then, \( (D^\alpha_{x}U)(t) > 0 \) for \( t \geq \tau_1 \). Otherwise, \( (D^\alpha_{x}U)(t) < 0 \) for \( t \geq \tau_1 \).
there exists $\tau_2 \in [\tau_1, \infty)$, such that $(D_{\ast}^\alpha, U)(\tau_2)v(\tau_2) < 0$. Since $(D_{\ast}^\alpha, U)(t)v(t)$ is strictly decreasing for $t \geq \tau_1$, so $(D_{\ast}^\alpha, U)(t)v(t) < (D_{\ast}^\alpha, U)(\tau_2)v(\tau_2) = C < 0$ for $t \geq \tau_2$. From Lemma 3.3, we have

$$
E(t) = (D_{\ast}^\alpha, U)(t) < C \exp(-\int_{\tau_2}^{t} b(s)ds), t \geq \tau_2.
$$

Integrating from $\tau_2$ to $t$, we have

$$
E(t) < E(\tau_2) + \Gamma(1-\alpha)C\int_{\tau_2}^{t} \exp(-\int_{\tau_2}^{s} b(\tau)d\tau)ds.
$$

Letting $t \to \infty$, and using condition (18), we get $\lim_{t \to \infty} E(t) = -\infty$, which is contradicts to the fact that $E(t) > 0$.

Hence $(D_{\ast}^\alpha, U)(t) > 0$ for $t \geq \tau_1$. Let $\omega(t) = (D_{\ast}^\alpha, U)(t)$. From Lemma 2.4 and impulsive fractional inequality (7) and (8), we have

$$
\omega(t) \leq -b(t)\omega(t) - G(t), t \geq t_0, t \neq t_k.
$$

By using Lemma 3.2, we have

$$
\omega(t) \leq \omega(\tau_1)\prod_{\tau \in ]t_0,t]}(1+\alpha_k)\exp(-\int_{\tau_1}^{t} b(s)ds)
$$

by using the condition (19) and taking $t \to \infty$, it follows from above equation

$$
\lim_{t \to \infty} \inf \prod_{\tau \in ]t_0,t]}(1+\alpha_k)\exp(-\int_{\tau_1}^{t} b(s)ds) = -\infty,
$$

which contradicts $\omega(t) > 0$. The proof is completed.

On the other hand, suppose to contrary that the impulsive fractional differential inequality (9) and (10) has an eventually negative solution $U(t)$, then there exists $t_0 \geq 0$ such that $U(t) < 0$, $U(t-\delta) < 0$ for $0, t \geq \tau_1$. Then using similar methods, according the condition (20) and taking $t \to \infty$, we can obtain

$$
\limsup_{t \to \infty} \prod_{\tau \in ]t_0,t]}(1+\alpha_k) \exp(-\int_{\tau_0}^{t} b(s)ds) = \infty,
$$

and

$$
\int_{\tau_0}^{t} \prod_{\tau \in ]t_0,t]}(1+\alpha_k) \exp(-\int_{\tau_0}^{t} b(s)ds) = \infty.
$$

In this section, we discuss an illustrative example.

Example 4.1 Consider the following problem

$$
\begin{aligned}
\left\{ 
\begin{array}{ll}
\frac{D_{\ast}^\frac{3}{4}}{t} u(t,x) + \frac{3}{4} D_{\ast}^\frac{3}{4} u(t,x) = e^{-u^2}(t,x)\Delta u(t,x) - \\
\left( 1 + t^2 + x^2 \right) u(t,x) - \left( \frac{2\pi}{3},x \right) e^{1 - \frac{2\pi x}{3}} - t \cos x, t \neq t_k,
\end{array}
\right.
\end{aligned}
$$

(21)

with the boundary condition

$$
\frac{\partial u(t,x)}{\partial n} = w(t,x,u(t,x)) = -u^3(t,x), (t,x) \in R_x \times \partial \Omega, t \neq t_k.
$$

(22)

Here

$$
\alpha = \frac{3}{4}, \Omega = (0, \frac{\pi}{2}), n = 1, b(t) = \frac{1}{t}, a(t) = e^{-t}, h(u) = u^3,
$$

$$
q(t,x) = 1 + t^2 + x^2, f_i(u) = \left( \frac{2\pi}{3},x \right) e^{1 - \frac{2\pi x}{3}}, g(t,x) = t \cos x,
$$

$$
\sigma(t_k,x) = t_k^3 \cos x, \alpha_k = k^3, (t,x) \in R_x \times \left( 0, \frac{\pi}{2} \right).
$$

We can easily see that

$$
\int_{\tau_0}^{t} \exp(-b(s)ds)ds = \int_{\tau_0}^{t} \exp(-\int_{\tau_0}^{s} b(\tau)d\tau)ds = \int_{\tau_0}^{t_0} \frac{1}{t} dt \int_{\tau_0}^{t} \frac{1}{t} dt = \infty,
$$

and

$$
\limsup_{t \to \infty} \prod_{\tau \in ]t_0,t]}(1+\alpha_k) \exp(-\int_{\tau_0}^{t} b(s)ds) = \infty.
$$
The conditions of Theorem 3.4 are satisfied, thus every solution of the problem (21)-(22) oscillates.

REFERENCES


