A New Relaxation Method for Binary Quadratic Programming: An Application to Densest k-subgraph

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Abstract—Binary quadratic programming (BQP) problem was an NP-hard problem and had a large number of applications. In this paper, a new relaxation method, that was doubly nonnegative relaxation, was proposed for solving BQP problem. Moreover, we prove that the doubly nonnegative relaxation for BQP is equivalent to a new tighter semidefinite relaxation. When BQP problem reduces to densest k-subgraph problem, the doubly nonnegative relaxation is equivalent to a tighter semidefinite relaxation. Finally, some comparative numerical results are reported to show that the efficiency of the doubly nonnegative relaxation is more promising than that of semidefinite relaxation for solving some specific BQP problems.

Keywords—binary quadratic programming; doubly nonnegative relaxation; semidefinite relaxation; densest k-subgraph

I. INTRODUCTION

In this paper, we consider the following binary quadratic programming (BQP) problem

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & a_i^T x = b_i, \quad i = 1, 2, K, m, \\ & x \in \{-1, 1\}^n, \end{aligned} \quad (\text{B}\Theta\text{II})$$

where $x \in R^n$ is the decision variable, $Q \in R^{n \times n}$, $c \in R^n$, $a_i \in R^n$ and $b_i \in R$ for all $i \in I := \{1, 2, \dots, m\}$ are the data. Without loss of generality, Q is symmetric, and we assume Q is not positive semidefinite, which implies generally that problem (BQP) is nonconvex and NP-hard.

Problem (BQP) arises in many applications, such as financial analysis [1], molecular conformation problem [2] and cellular radio channel assignment [3]. Many combinatorial optimization problems are special cases of problem (BQP), such as max-cut (MC) problem [4], dense k-subgraph (DkS) problem [5]. For solving this type of problem, a systematic survey of the solution methods can be found in [6] of Chap. 10, and the references therein.

It is well-known that semidefinite relaxation is a powerful and computationally efficient approximation technique for solving a host of very difficult optimization problems, for instance, MC problem, boolean quadratic programming problem. It has also been at the center of some of the very exciting developments in the area of signal processing and

communications [7]. The standard semidefinite relaxation for problem (BQP) is

$$\begin{aligned} \min \quad & X \bullet Q + 2c^T x \\ \text{s.t.} \quad & a_i^T x = b_i, \quad \forall i \in I, \\ (\text{SDR}) \quad & a_i^T X a_i = b_i^2, \quad \forall i \in I, \\ & X_{ii} = 1, \quad \forall i = 1, 2, K, n, \\ & X \in S_n^+, \end{aligned}$$

It is obviously that problem (SDR) is convex and gives a lower bound for problem (BQP) if the feasible set of problem (BQP) is nonempty. Moreover, if an optimal solution (x^*, X^*) of problem (SDR) satisfies $X^* = x^*(x^*)^T$, then we can conclude that x^* is also an exact optimal solution for problem (BQP).

Recently, Burer [11] presents a computable convex problem, i.e., doubly nonnegative programming which can be efficiently solved by some popular packages, to solving a large class of NP-hard nonconvex quadratic programs. For more details and developments of this technique, one may refer to [8-11] and the references therein.

Inspired by the idea in [11], the doubly nonnegative relaxation and the new semidefinite relaxation for problem (BQP) are established, respectively. We prove that the doubly nonnegative relaxation for problem (BQP) is equivalent to the new semidefinite relaxation. Applying this result to DkS problem, we obtain that the doubly nonnegative relaxation for DkS problem is equivalent to the new semidefinite relaxation. Moreover, some comparative numerical results are reported to illustrate the efficiency of doubly nonnegative relaxation and semidefinite relaxation, respectively.

II. NEW RELAXATION FOR PROBLEM (BQP)

For simplicity, let $A := (a_1, a_2, \dots, a_m)^T$, $b := (b_1, b_2, \dots, b_m)$, S_n denotes the cone of symmetric matrices, S_n^+ the cone of $n \times n$ positive semidefinite matrices, N_n the cone of $n \times n$ nonnegative symmetric matrices and e the vector of ones with appropriate dimension. $A \bullet B = \text{trace}(A^T B)$ for two conformal matrices A and B . $\text{diag}(M)$ is a column vector whose elements are the diagonal elements of the given matrix M .

A. Doubly Nonnegative Relaxation for Problem (BQP)

Let $z = (e - x) / 2$ in problem (BQP), then $z \in \{0, 1\}^n$, and problem (BQP) can be equivalently reformulated as

$$\begin{aligned} \min \quad & 4z^T Qz - 4z^T (Qe + c) + e^T Qe + 2c^T e \\ \text{(R-BQP)} \quad & s.t. \quad 2Az = Ae - b, \\ & z \in \{0, 1\}^n. \end{aligned}$$

According to Theorem 2.6 in [8], problem (R-BQP) can be further equivalently transformed into the following completely positive programming problem, where CP_{1+n} is defined as follows

$$CP_{1+n} := \{X \in S_{1+n} : X = \sum_{k \in K} z^k (z^k)^T\}$$

for some finite $\{z^k\}_{k \in K} \subset R_{1+n}^+$, where CP_{1+n} is a closed convex cone, which is called completely positive cone. Thus problem (CPP) is a convex problem. However, problem (CPP) is NP-hard, since checking whether or not a given matrix belongs to CP_{1+n} is NP-hard [12]. Thus, it has to be replaced by some computable cones, which can efficiently approximate by CP_{1+n} .

$$\begin{aligned} \min \quad & 4Q \bullet Z - 4z^T (Qe + c) + e^T Qe + 2c^T e \\ \text{(CPP)} \quad & s.t. \quad 2Az = Ae - b, \\ & 4\text{diag}(AZA^T) = \text{diag}((Ae - b)(Ae - b)^T), \\ & \text{diag}(Z) = z, \\ & \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} \in CP_{1+n}. \end{aligned}$$

Note that the convex cone N_n is self-dual, and so is the convex cone S_n^+ . Hence, Diananda's decomposition theorem [13] can be reformulated as follows.

Theorem 1 $CP_n \subseteq S_n^+ \cap N_n$ holds for all n . If $n \leq 4$, then $CP_n = S_n^+ \cap N_n$.

By the way, the matrices in $S_n^+ \cap N_n$ sometimes are called doubly nonnegative. Of course, in dimension $n \geq 5$ there are matrices which are doubly nonnegative but not completely positive, the counterexample can be seen in [14]. By using Theorem 1, problem (CPP) can be relaxed to the following doubly nonnegative programming problem

$$\begin{aligned} \min \quad & 4Q \bullet Z - 4z^T (Qe + c) + e^T Qe + 2c^T e \\ \text{(DNNP)} \quad & s.t. \quad 2Az = Ae - b, \\ & 4\text{diag}(AZA^T) = \text{diag}((Ae - b)(Ae - b)^T), \\ & \text{diag}(Z) = z, \\ & \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} \in S_{1+n}^+ \cap N_{1+n}, \end{aligned}$$

which is computable by some popular packages, such as CVX. Note that problem (DNNP) is similar to the semidefinite relaxation for problem (R-BQP), the only difference is exist in the last conic constraint. In the following subsection, problem (DNNP) will be compared with some semidefinite relaxation problems for problem (BQP) for solving some (BQP) instances.

B. New Tighter Semidefinite Relaxation for Problem (BQP)

Note that problem (BQP) can also be relaxed to the following problem

$$\begin{aligned} \min \quad & Q \bullet X + 2c^T x \\ \text{(R-SDR)} \quad & s.t. \quad Ax = b, \\ & \text{diag}(AXA^T) = \text{diag}(b(b)^T), \\ & \text{diag}(X) = e, \\ & X - xx^T \in S_n^+. \end{aligned}$$

If the optimal solution (x^*, X^*) of problem (R-SDR) satisfies $X^* = x^* (x^*)^T$, it holds that x^* is also an optimal solution of problem (BQP). On one hand, it is worth noting that

$$X - xx^T \in S_n^+ \Rightarrow X \in S_n^+ \quad (1)$$

always true, which implies that any feasible solution of problem (R-SDR) is also feasible for problem (SDR), i.e., $\text{Opt}(\text{SDR}) \leq \text{Opt}(\text{R-SDR})$ since the two problems have the same objective functions, where $\text{Opt}(\bullet)$ denotes optimal value for corresponding objective function. Therefore, we can conclude that problem (R-SDR) is the tighter semidefinite relaxation for problem (BQP) than problem (SDR).

On the other hand, we can easily verify that the constraint $X - xx^T \in S_n^+$ is nonconvex, since the quadratic term $-xx^T$ is nonconvex. Thus, problem (R-SDR) is nonconvex and can not be solved by some popular packages for solving convex programs. In order to establish the convex representation for problem (R-SDR), a crucial theorem is given below and the details of its proof can be seen in Appendix A.5.5 Schur complement in [15].

Theorem 2 Let $M \in S_n$ is partitioned as $M = \begin{bmatrix} D & B \\ B^T & C \end{bmatrix}$. If $\det(D) \neq 0$, then the following relations hold, where $H = C - B^T D^{-1} B$,

- (i) $M \preceq 0$ if and only if $D \preceq 0$ and $H \preceq 0$;
(ii) If $D \preceq 0$, then $M \in S_n^+$ if and only if $H \in S_n^+$.

By Theorem 2(ii) and (1), it holds immediately that

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in S_{1+n}^+ \Leftrightarrow X - xx^T \in S_n^+ \Rightarrow X \in S_n^+. \quad (2)$$

So, problem (R-SDR) is equivalently reformulated as follows

$$\begin{aligned} \min \quad & Q \bullet X + 2c^T x \\ \text{s.t.} \quad & Ax = b, \\ \text{(SDR-1)} \quad & \text{diag}(AXA^T) = \text{diag}(b(b)^T), \\ & \text{diag}(X) = e, \\ & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in S_{1+n}^+. \end{aligned}$$

which can be efficiently solved by some popular packages for solving convex programs. Here, some examples are tested to show that problem (SDR1) is a tighter relaxation for problem (BQP) compared with problem (SDR), and the corresponding numerical results also show that problem (SDR1) is more promising. These examples are solved by CVX [16], a package for specifying and solving convex programs.

Example 1 This is a two-dimensional nonconvex problem with one linear equality constraint, and the corresponding coefficients are given as follows

$$Q = \begin{bmatrix} 0 & -3 \\ -3 & -20 \end{bmatrix}, c = \begin{bmatrix} -8 \\ 9 \end{bmatrix}, A = [10, -10], b = 0.$$

On one hand, by using CVX to solve problem (SDR), we obtain $\text{Opt}(\text{SDR}) = -\infty$, since problem (SDR) is unbounded below. On the other hand, when problem (SDR1) is solved, one has $\text{Opt}(\text{SDR1}) = -28$ with $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = [-1, -1]^T$.

Note that the relationship $X = xx^T$ holds, thus we can conclude that $x = [-1, -1]^T$ is also an optimal solution for problem (BQP). The results show that problem (SDR1) is more tight and efficient than problem (SDR) for solving this problem.

Example 2 The five-dimensional problem with three linear equality constraints, and the corresponding coefficients are chosen as follows

$$Q = \begin{bmatrix} -52 & 31 & 49 & -7 & 4 \\ 31 & -16 & -50 & -13 & -49 \\ 49 & -50 & 8 & 44 & -30 \\ -7 & -13 & 44 & 36 & 12 \\ 4 & -49 & -30 & 12 & 56 \end{bmatrix}, c = \begin{bmatrix} -20 \\ 37 \\ 43 \\ 25 \\ -6 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 10 & 29 & 14 & -36 \\ 38 & 9 & 1 & -17 & 23 \\ 48 & 39 & 5 & -17 & -13 \end{bmatrix}, b = \begin{bmatrix} 11 \\ -50 \\ -36 \end{bmatrix}.$$

If this problem is solved by problem (SDR), then it returns $\text{Opt}(\text{SDR}) = -\infty$ since problem (SDR) is unbounded below. When using problem (SDR1) to solve this problem, we have $\text{Opt}(\text{SDR1}) = -307.548$. However, the relationship $X = xx^T$ does not hold. Therefore, we obtain a tighter lower bound -307.548. These results also show that problem (SDR1) is more effective than problem (SDR).

In fact, the constraint $x \in \{-1, 1\}^n$ in problem (BQP) implies

$$\begin{aligned} \min \quad & Q \bullet X + 2c^T x \\ \text{s.t.} \quad & Ax = b, \\ & \text{diag}(AXA^T) = \text{diag}(b(b)^T), \\ \text{(SDR2)} \quad & \text{diag}(X) = e, \\ & 1 - x_i - x_j + X_{ij} \geq 0, \quad \forall 1 \leq i \leq j \leq n, \\ & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in S_{1+n}^+. \end{aligned}$$

that

$$(1 - x_i)(1 - x_j) \geq 0 \Rightarrow 1 - x_i - x_j + X_{ij} \geq 0 \quad (3)$$

for all $1 \leq i \leq j \leq n$. Substituting (3) into problem (SDR1), we can get the new semidefinite relaxation problem (SDR2).

Note that each feasible solution of problem (SDR2) is also feasible for problem (SDR1). Thus the lower bound of problem (SDR2) is more tight than that of problem (SDR1), since $n(n+1)/2$ new inequality constraints are added into the corresponding problem (SDR2).

Now, we test some nonconvex instances to show that the lower bound of problem (SDR2) is tighter than that of problem (SDR1) from the computational point of view. Moreover, it is worth noting that problem (DNNP) has $n + n(n+1)/2$ equality constraints more than problem (SDR1), and n equality constraints more than problem (SDR2), respectively. Thus, the lower bound which obtained from problem (DNNP) is much greater than that one from problem (SDR1) and problem (SDR2), respectively. Two types of instances are tested.

TABLE I. STATISTICS OF THE TEST INSTANCES

Name	Instances	n/m	Functions
RdBQP	50	50/25	$Q=\text{rand}(n,n)+\text{rand}(n,n)^T$, $c=\text{rand}(n,1)$, $A=\text{rand}(m,n)$, $b=\text{rand}(m,1)$
RdsBQP	50	50/25	$Q=\text{rands}(n,n)+\text{rands}(n,n)^T$, $c=\text{rands}(n,1)$, $A=\text{rands}(m,n)$, $b=\text{rands}(m,1)$

FIGURE I. THE PERFORMANCE OF LEFT FIGURE IS BASED ON OPTIMAL VALUES OF RDBQP, AND THE PERFORMANCE OF RIGHT FIGURE IS BASED ON OPTIMAL VALUES OF RDSBQP

To compare the performance of problems (SDR1), (SDR2) with that of problem (DNNP) for solving problem (BQP), we use the performance profiles described in Dolan and More's paper [17]. These instances are solved by CVX, and the results of performance are shown in FIGURE I. From FIGURE I, it is obviously that the performances of problem (SDR2) and problem (DNNP) are almost the same, which are better than that one of problem (SDR1), for solving RdBQP and RdsBQP, respectively. Thus, we can obtain that it is more efficient to use problem (SDR2) and problem (DNNP) for solving problem (BQP) than to use problem (SDR1), from the point of view of optimal values.

III. THE EQUIVALENCE BETWEEN RELAXATION PROBLEMS

In this section, we investigate the relationship between the two relaxation problems (DNNP) and (SDR2). First of all, the definition of the equivalence of two optimization problems is given as follows.

Definition 1 Two problems are called equivalent if they satisfy the following two conditions:

- (i) If from a solution of one problem, a solution of the other problem is readily found, and vice versa.
- (ii) The two problems have the same optimal objective value.

Now, based on the above Definition 1, the main theorem is given below.

Theorem 3. Suppose that the feasible sets $\text{Fea}(\text{DNNP})$ and $\text{Fea}(\text{SDR2})$ are both nonempty. Then two problems (DNNP) and (SDR2) are equivalent.

Proof: The proof can be divided into two parts. First of all, we will prove that $\text{Opt}(\text{SDR2}) \geq \text{Opt}(\text{DNNP})$.

Suppose that (x^*, X^*) is an optimal solution of problem (SDR2), let

$$Z = (ee^T - e(x^*)^T - x^*e^T + X^*) / 4, z = (e - x^*) / 2. \quad (4)$$

By $Ax^* = b$ and (4), we have

$$Ax^* = A(e - 2z) = b \Rightarrow 2Az = Ae - b. \quad (5)$$

From (4) and $\text{diag}(AX^*A) = \text{diag}(bb^T)$, it follows that

$$\begin{aligned} \text{diag}(AX^*A) &= \text{diag}(A(4Z - ee^T + e(x^*)^T + x^*e^T)A^T) \quad \text{i.e.,} \\ &= \text{diag}(bb^T), \end{aligned}$$

$$4\text{diag}(AZA^T) = \text{diag}((Ae - b)(Ae - b)^T). \quad (6)$$

Again from (4), one has

$$\text{diag}(Z) = (e - 2x^* + \text{diag}(X^*)) / 4 = z, \quad (7)$$

since $\text{diag}(X^*) = e$.

From $1 - x_i^* - x_j^* + X_{ij}^* \geq 0, \forall 1 \leq i \leq j \leq n$, it holds that

$$Z_{ij} \geq 0, \forall 1 \leq i \leq j \leq n, \quad (8)$$

which combining with (7), it is true that

$$z_i \geq 0, \forall 1 \leq i \leq n. \quad (9)$$

By Theorem 2(ii) and (4), it follows that

$$\begin{aligned} Z - zz^T &= (ee^T - e(x^*)^T - x^*e^T + X^*) / 4 \\ &\quad - (e - x^*)(e - x^*)^T / 4 \\ &= (X^* - x^*(x^*)^T) \in S_n^+. \end{aligned} \quad (10)$$

Combine (5), (6), (8), (9) with (10), it follows that (z, Z) defined by (4) is a feasible solution for problem (DNP).

Moreover, again from (4), we have

$$\begin{aligned} 4Q \bullet Z - 4z^T(Qe + c) + e^T Qe + 2c^T e \\ = Q \bullet (ee^T - e(x^*)^T - x^*e^T + X^*) - 2(e - x^*)^T(Qe + c) \\ + e^T Qe + 2c^T e \\ = Q \bullet X^* + 2c^T x^* = \text{Opt}(\text{SDR2}), \end{aligned}$$

which further implies that $\text{Opt}(\text{DNNP}) \leq \text{Opt}(\text{SDR2})$.

On the other hand, given an optimal solution (z^*, Z^*) of problem (DNNP), and let

$$X_{ij} = 1 - 2z_i^* - 2z_j^* + 4Z_{ij}^*, x_i = 1 - 2z_i^*, \quad (11)$$

for all $1 \leq i \leq j \leq n$, which imply that

$$\text{diag}(X) = e - 4z^* + 4\text{diag}(Z^*) = e, \quad (12)$$

since $\text{diag}(Z^*) = z^*$. Moreover, for $1 \leq i \leq j \leq n$,

$$\begin{aligned} 1 - x_i - x_j + X_{ij} &= 1 - (1 - 2z_i^*) - (1 - 2z_j^*) + 1 - 2z_i^* \\ &\quad - 2z_j^* + 4Z_{ij}^* \\ &= 4Z_{ij}^* \geq 0. \end{aligned} \quad (13)$$

From (11) and $2Az^* = Ae - b$, it follows that

$$Ax = A(e - 2z^*) = b. \quad (14)$$

Again from (11), we have

$$\begin{aligned} \text{diag}(AXA^T) &= \text{diag}(A(ee^T - 2e(z^*)^T - 2z^*e^T + 4Z^*)A^T) \\ &= \text{diag}(bb^T). \end{aligned} \quad (15)$$

From (11) and Theorem 2.2(ii), it holds that

$$\begin{aligned} X - xx^T &= ee^T - 2z^*e^T - 2e(z^*)^T + 4Z^* \\ &\quad - (e - 2z^*)(e - 2z^*)^T \\ &= 4(Z^* - z^*(z^*)^T) \in S_n^+. \end{aligned} \quad (16)$$

By (13), (14), (15) and (16), we can conclude that (x, X) defined by (11) is a feasible solution for problem (SDR2). Furthermore, we have

$$\begin{aligned} X \bullet Q + 2c^T x &= (ee^T - 2e(z^*)^T - 2z^*e^T + 4Z^*) \bullet Q \\ &\quad + 2c^T(e - 2z^*) \\ &= 4Z^* \bullet Q - 4(z^*)^T(Qe + c) + e^T Qe + 2c^T e \\ &= \text{Opt}(\text{DNNP}), \end{aligned}$$

which implies that $\text{Opt}(\text{SDR2}) \leq \text{Opt}(\text{DNNP})$.

Up to now, we prove that $\text{Opt}(\text{SDR2}) = \text{Opt}(\text{DNNP})$. By (4) and (11), it holds that optimal solutions of problem (SDR2) and problem (DNNP) can be representation each other. According to Definition 3.1, we can conclude that problem (DNNP) is equivalent to problem (SDR2).

FIGURE II. THE PERFORMANCE OF LEFT FIGURE IS BASED ON THE NUMBER OF ITERATIONS OF RDBQP, AND RIGHT FIGURE IS BASED ON THE NUMBER OF ITERATIONS OF RdsBQP

Although $\text{Opt}(\text{SDR2}) = \text{Opt}(\text{DNNP})$ in view of Theorem 3 and Definition 1, problem (DNNP) has n equality constraints more than problem (SDR2) in form. So, the amount of computation for solving problem (DNNP) may be much greater than that of for solving problem (SDR2). In order to illustrate this point of view, the compared performance results are shown in FIGURE II, which is based on the number of iterations for solving RdbQP and RdsBQP. The results in FIGURE II show that the performance of problem (SDR2) is better than that of problem (DNNP) for solving RdbQP, but the performance of problem (DNNP) is better than that of problem (SDR2) for solving RdsBQP, from the point of view of the number of iterations.

IV. AN APPLICATION: DENSEST k -SUBGRAPH PROBLEM

The densest k -subgraph (DkS) problem [5] takes as input a graph G on n vertices and a parameter k , and asks for a subgraph of G on at most k vertices having the maximum total weight of edges. This problem is NP-hard and is a fundamental graph optimization problem and arises in several applications, such as community detection in social networks and identifying protein families [5], etc.

Given a symmetric $n \times n$ matrix $A = (a_{ij})$. Let us associate a weighted graph with vertex set $\{1, K, n\}$ to A in the usual way: If $i < j$ and $a_{ij} \neq 0$, we introduce the edge $[i, j]$ with weight a_{ij} in the graph. Thus we interpret A as the weighted adjacency matrix of a graph with vertex set $V = \{1, K, n\}$. The goal of DkS problem is to find a subset V_I of V consisting of k vertices, such that the total weight of edges in the subgraph spanned by V_I is maximized. This leads to DkS problem has the following formulation

$$\begin{aligned} \max \quad & y^T A y \\ \text{(DkS)} \quad & \text{s.t.} \quad y^T e = k, \\ & y \in \{0, 1\}^n. \end{aligned}$$

By using the techniques introduced in section 2, we can get doubly nonnegative relaxation and standard semidefinite relaxation problems for problem (DkS) as

$$\begin{aligned}
 & \max \quad A \bullet Y / 2 \\
 & s.t. \quad y^T e = k, \\
 & \quad \quad e^T Y e = k^2, \\
 & \quad \quad \text{diag}(Y) = y, \\
 & \quad \quad \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \in S_{1+n}^+ \cap N_{1+n},
 \end{aligned}
 \tag{DNNP}_{DkS}$$

and

$$\begin{aligned}
 & \max \quad A \bullet Y / 2 \\
 & s.t. \quad y^T e = k, \\
 & \quad \quad e^T Y e = k^2, \\
 & \quad \quad \text{diag}(Y) = y, \\
 & \quad \quad \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \in S_{1+n}^+,
 \end{aligned}
 \tag{I-SDR}_{DkS}$$

respectively. Moreover, a new tighter semidefinite relaxation problem could be established for problem (DkS) as following

$$\begin{aligned}
 & \max \quad A \bullet Z / 8 + z^T A e / 4 + e^T A e / 8 \\
 & s.t. \quad z^T e = 2k - n, \\
 & \quad \quad e^T Z e = (2k - n)^2, \\
 & \quad \quad \text{diag}(Z) = e, \\
 & \quad \quad 1 + z_i + z_j + Z_{ij} \geq 0, \forall 1 \leq i \leq j \leq n, \\
 & \quad \quad \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} \in S_{1+n}^+.
 \end{aligned}
 \tag{II-SDR}_{DkS}$$

Similar to the analysis of Theorem 3, we can obtain the following theorem immediately.

Theorem 4 Suppose that feasible sets $\text{Fea}(\text{DNNP}_{DkS})$ and $\text{Fea}(\text{II-SDR}_{DkS})$ are nonempty. Then the two problems (DNNP_{DkS}) and (II-SDR_{DkS}) are equivalent.

To give a flavour of the actual behaviour of the three relaxed problems for solving problem (DkS), some instances in TABLE II are tested.

TABLE II. STATISTICS OF THE TEST INSTANCES

Name	Functions
A25	25/5 seed=1,2,...,50, rand('seed', seed), A=round(10*rand(n)), A25=tril(A, -1)+triu(A', 0)
A30	30/10 seed=2012, randn('seed', seed), A=randn(n,n), A=tril(A, -1)+triu(A', 0)
A40	40/15 seed=2017, randn('seed', seed), A=randn(n,n), A=tril(A, -1)+triu(A', 0)
A50	50/10 seed=2001, ..., 2050, randn('seed', seed), G=(rand(n)<0.5), A=round(20*rand(n)-10), A=tril(A, 1), A=A.*G, A50=A+A'
A60	60/20 seed=2020, randn('seed', seed), A=round(randn(n,n)), A=tril(A, -1)+triu(A', 0)

FIGURE III. THE PERFORMANCE PROFILE IS BASED ON THE NUMBER OF ITERATIONS OF PROBLEMS (DNNP_{DkS}) , (II-SDR_{DkS}) AND (I-SDR_{DkS}) FOR SOLVING A25 AND A50

The performance results for solving A25 and A50 are shown in FIGURE III. In view of the number of iterations, the results in FIGURE III show that the performance of problem (I-SDR_{DkS}) is better than that of problems (DNNP_{DkS}) and (II-SDR_{DkS}) for solving A25 and A50. According to Theorem 4, problems (DNNP_{DkS}) and (II-SDR_{DkS}) can give more tight bounds for A25 and A50, while the number of iterations is a bit more. Note that the performance of problem (II-SDR_{DkS}) is better than that of problem (DNNP_{DkS}) in the viewpoint of the number of iterations, for A25 and A50.

In TABLE III, we report the corresponding comparative numerical results of three test instances from the viewpoint of optimal values and the number of iterations, respectively. The results show that the efficiency of problem (DNNP_{DkS}) is better than that of problem (II-SDR_{DkS}) . Thus we can conclude that it may be more promising to use problem (DNNP_{DkS}) than to use problem (II-SDR_{DkS}) for solving some specific (DkS) problems in practice.

TABLE III. NUMERICAL RESULTS OF INSTANCES

	Model	Values	Number of iterations	CPU time
A30	(DNNP_{DkS})	30.5894	52	3.4
	(II-SDR_{DkS})	30.5903	70	5.0
	(I-SDR_{DkS})	33.3382	33	0.7
A40	(DNNP_{DkS})	74.4657	50	14.7
	(II-SDR_{DkS})	74.4658	66	20.6
	(I-SDR_{DkS})	75.5198	33	1.5
A60	(DNNP_{DkS})	150.385	52	105.5
	(II-SDR_{DkS})	150.387	60	125.3
	(I-SDR_{DkS})	154.272	40	3.2

V. CONCLUDING REMARKS

In this paper, a class of problems (BQP) is considered, which is NP-hard in general. In order to solve this class of problems, two types of convex relaxation methods are proposed. One of the methods is semidefinite relaxation. The other method is doubly nonnegative relaxation. Moreover, the two convex relaxation problems are equivalent. Applying these results to problem (DkS), we have the doubly nonnegative relaxation for problem (DkS) is equivalent to a new tighter semidefinite relaxation. Furthermore, the reported comparative numerical results show that the efficiency of doubly nonnegative relaxation is more promising than that of semidefinite relaxation for solving some specific (BQP) problems.

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