A new Hermite-Hadamard Type Inequality

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Abstract—In this paper, a better estimate of the Hermite-Hadamard integral inequality for the product of two convex functions is established, and the proof is given. Then the inferences and applications of the inequalities obtained are provided.

Keywords—Hermite-Hadamard integral inequality; Convex functions

I. INTRODUCTION

The following definition is well-known in the literature. A real-valued function \( f : [a, b] \to \mathbb{R} \), \( \emptyset \neq I = [a, b] \subseteq \mathbb{R} \), is said to be convex on \([a, b]\) if inequality

\[
(1-t)f(x) + tf(y) \leq f(tx + (1-t)y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \). Conversely, if the opposite inequality holds, the function is said to be concave on \( I \). A function \( f \) is convex on \( I \) if and only if \( f''(x) \geq 0 \) for all \( x \in (a, b) \). Let \( f : [a, b] \to \mathbb{R} \) be a convex mapping. The following inequality

\[
1 \leq \frac{b-a}{2} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

is known in the literature as Hadamard’s inequality for convex mapping.

So far, a lot of literatures have been researched on the famous Hermite-Hadamard inequality which was first published in [1]. In recent years, some scholars have tried to improve and generalize the classical inequality. For example, ABDALLAH EL FARISSI had given new estimates in [2]; In [3], Hadamard inequality is concerned with the compound of two convex functions, which is proposed by Xiang Gao.

II. MAIN RESULTS

In this part, we will establish a new estimate of the product of two convex functions, which is a generalization of the following inequality:

**THEOREM 1.**[3]

Let \( f, g : [a, b] \to [0, \infty) \) be convex functions on \([a, b] \subseteq \mathbb{R} , a < b \). Then

\[
1 \leq \frac{b-a}{2} \int_a^b f(x)g(x)dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)
\]

and

\[
2f(\frac{a+b}{2})g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b)
\]

where

\[
M(a, b) = f(a)g(a) + f(b)g(b)
\]

\[
N(a, b) = f(a)g(b) + f(b)g(a)
\]

According to the Theorem1, we can easily gain the following results.

**COROLLARY 1.**

Let \( f, g : [a, b] \to [0, \infty) \) be convex functions on \([a, b] \subseteq \mathbb{R} \) . And then meet \([f(a) - f(b)], [g(a) - g(b)] \leq 0 \), we can learn that the product of \( f \) and \( g \) is still satisfied (1) and (2).

Secondly, we get the following main result.

**THEOREM 2.**

Let \( f, g : [a, b] \to [0, \infty) \) be convex functions on \([a, b] \subseteq \mathbb{R} \) , and meet \([f(a) - f(b)], [g(a) - g(b)] \leq 0 \) . Then for all \( \lambda \in [0,1] \), we have

\[
1 \leq \frac{b-a}{2} \int_a^b f(x)g(x)dx \leq L(\lambda) \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)
\]

and

\[
2f(\frac{a+b}{2})g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b)
\]

where

\[
L(\lambda) = \frac{1}{b-a} \int_a^b f(x)g(x)dx
\]
\[
L(\lambda) = \frac{\lambda}{3} f(a)g(a) + \frac{1 - \lambda}{3} f(b)g(b) \\
\quad + \frac{1}{3} f(\lambda b + (1 - \lambda)a)g(\lambda b + (1 - \lambda)a) \\
\quad + \frac{1}{6} f(\lambda b + (1 - \lambda)a)[\lambda g(a) + (1 - \lambda)g(b)] \\
\quad + \frac{1}{6} g(\lambda b + (1 - \lambda)a)[\lambda g(b) + (1 - \lambda)g(a)]
\]

and

\[
l(\lambda) = 2\lambda f\left(\frac{2 - \lambda}{2}a + \frac{\lambda}{2}b\right)g\left(\frac{2 - \lambda}{2}a + \frac{\lambda}{2}b\right) \\
\quad - \frac{1 + 3\lambda - 3\lambda^2}{6} M(a,b) - \frac{2 + 3\lambda - 3\lambda^2}{6} N(a,b) \\
\quad + 2(1 - \lambda)f\left(\frac{1 - \lambda}{2}a + \frac{1 + \lambda}{2}b\right)g\left(\frac{1 - \lambda}{2}a + \frac{1 + \lambda}{2}b\right)
\]

**Corollary 2.**

If replace “ \([f(a) - f(b)],[g(a) - g(b)] \leq 0 \)” with neither \(f\) nor \(g\) are increasing or decreasing synchronously. Then for all \(\lambda \in [0,1]\), the results of Theorem 2 are still satisfied.

**Corollary 3.**

Let \(f, g : [a,b] \to [0,\infty)\) be convex functions on \([a,b] \subseteq I\), \(a < b\), and meet \([f(a) - f(b)],[g(a) - g(b)] \leq 0\).
Then for all \(\lambda \in [0,1]\), we have the following inequality

\[
\frac{1}{b - a} \int_a^b f(x)g(x)dx \leq \inf_{\lambda \in [0,1]} L(\lambda) \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)
\]

and

\[
2 f\left(\frac{a + b}{2}\right)g\left(\frac{a + b}{2}\right) - \frac{1}{6} M(a,b) - \frac{1}{3} N(a,b) \\
\leq \sup_{\lambda \in [0,1]} l(\lambda) \leq \frac{1}{b - a} \int_a^b f(x)g(x)dx
\]

where \(l(\lambda), L(\lambda)\) are defined in Theorem 2.

**Remark 1.** Applying Theorem 2 for \(\lambda = 1\), we get inequality (1) and (2). In addition, if we choose \(g(x) = 1\), then we have the right side of Hermite-Hadamard inequality.

### III. Lemma

In order to prove Theorem 2, we shall need the following Lemma:

**Lemma 1.**

Let \(F\) be convex functions on \([a,b] \subseteq I\), \(a < b\), Then for all \(\lambda \in [0,1]\), we have

\[
F\left(\frac{a + b}{2}\right) \leq \lambda F\left(\frac{\lambda b + (2 - \lambda)a}{2}\right) + (1 - \lambda) F\left(\frac{(1 + \lambda)b + (1 - \lambda)a}{2}\right)
\]

**Proof. Let**

\[
a + b = \lambda \frac{\lambda b + (2 - \lambda)a}{2} + (1 - \lambda) \frac{(1 + \lambda)b + (1 - \lambda)a}{2}
\]

Then, applying the properties of the convex function, (5) can be proved easily.

### IV. Proof of the Theorems

**Proof of Theorem 2.**

Firstly, discuss about (3): Applying (1) on the subinterval \([a, \lambda b + (1 - \lambda)a]\), with \(\lambda \neq 0\), we have

\[
\frac{1}{\lambda(b-a)} \int_a^{\lambda b+(1-\lambda)a} f(x)g(x)dx
\]

\[
\leq \frac{1}{3} [f(\lambda(a) + f(\lambda(b+(1-\lambda)a)]g(\lambda(b+(1-\lambda)a]) + \frac{1}{6} [f(a)g(\lambda(b+(1-\lambda)a) + f(\lambda(b+(1-\lambda)a)g(a)]
\]

Applying (1) on the subinterval \([\lambda b + (1 - \lambda)a,b]\), with \(\lambda \neq 1\), we have

\[
\frac{1}{(1-\lambda)(b-a)} \int_{\lambda b+(1-\lambda)a}^{b} f(x)g(x)dx
\]

\[
\leq \frac{1}{3} [f(\lambda b+(1-\lambda)a)g(\lambda b+(1-\lambda)a) + f(b)g(b)] + \frac{1}{6} [f(\lambda b+(1-\lambda)a)g(b) + f(b)g(\lambda b+(1-\lambda)a)]
\]

Multiplying (6) by \(\lambda\), (7) by \((1-\lambda)\), then adding the inequalities, we get:

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx
\]

\[
\leq \frac{\lambda}{3} f(a)g(a) + \frac{1 - \lambda}{3} f(b)g(b) \\
\quad + \frac{1}{3} [\lambda f(b) + (1 - \lambda) f(a)][\lambda g(b) + (1 - \lambda) g(a)] \\
\quad + \frac{1}{6} [\lambda f(b) + (1 - \lambda) f(a)][\lambda g(b) + (1 - \lambda) g(b)] \\
\quad + \frac{1}{6} [\lambda g(b) + (1 - \lambda) g(a)][\lambda g(b) + (1 - \lambda) g(a)]
\]

\[
= \frac{1}{3} [f(a)g(a) + f(b)g(b)] + \frac{1}{6} [f(b)g(a) + f(a)g(b)]
\]
\begin{align*}
\frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)
\end{align*}

So, we get that \( L(\lambda) \) is defined as in Theorem 2.

Secondly, discuss about (4):
According to (4), it also can be written as

\begin{align*}
\frac{1}{b - a} \int_{a}^{b} f(x) g(x) dx & \geq 2 f\left(\frac{a + b}{2}\right) g\left(\frac{a + b}{2}\right) - \frac{1}{6} M(a, b) - \frac{1}{3} N(a, b)
\end{align*}

Using the same method of above, we get

\begin{align*}
\frac{1}{b - a} \int_{a}^{b} f(x) g(x) dx & \geq 2 \lambda f\left(\frac{2 - \lambda}{2} a + \lambda b\right) g\left(\frac{2 - \lambda}{2} a + \lambda b\right) \\
& - \frac{\lambda}{6} \left[ f(a) g(a) + f(\lambda b + (1 - \lambda) a) g(\lambda b + (1 - \lambda) a)\right] \\
& - \frac{\lambda}{3} \left[ f(a) g(\lambda b + (1 - \lambda) a) + f(\lambda b + (1 - \lambda) a) g(a)\right] \\
& + 2(1 - \lambda) f\left(\frac{1 - \lambda}{2} a + (1 + \lambda) b\right) g\left(\frac{1 - \lambda}{2} a + (1 + \lambda) b\right) \\
& - \frac{1 - \lambda}{6} \left[ f(\lambda b + (1 - \lambda) a) g(\lambda b + (1 - \lambda) a) + f(\lambda b + (1 - \lambda) a) g(a)\right] \\
& - 2 \lambda f\left(\frac{1}{2} a + \frac{1}{2} b\right) g\left(\frac{1}{2} a + \frac{1}{2} b\right) \\
& - 2(1 - \lambda) f\left(\frac{1}{2} a + \frac{1}{2} b\right) g\left(\frac{1}{2} a + \frac{1}{2} b\right) \\
& + 2(1 - \lambda) f\left(\frac{1}{2} a + \frac{1}{2} b\right) g\left(\frac{1}{2} a + \frac{1}{2} b\right) \\
& - \frac{1 - \lambda}{6} \left[ f(\lambda b + (1 - \lambda) a) g(\lambda b + (1 - \lambda) a) + f(\lambda b + (1 - \lambda) a) g(a)\right] \\
& = 2 f(\cos^2 \theta) f\left(\frac{a(1 + \sin^2 \theta) + b \cos^2 \theta}{2}\right) g\left(\frac{a(1 + \sin^2 \theta) + b \cos^2 \theta}{2}\right) \\
& + 2\sin^2 \theta f\left(\frac{a \sin^2 \theta + b(1 + \cos^2 \theta)}{2}\right) g\left(\frac{a \sin^2 \theta + b(1 + \cos^2 \theta)}{2}\right) \\
& - \frac{1 + 3 \cos^2 \theta - 3 \cos^4 \theta}{6} M(a, b) - \frac{2 - 3 \cos^2 \theta + 3 \cos^4 \theta}{6} N(a, b)
\end{align*}

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REFERENCES

EXAMPLE.
Let \( \lambda = \cos \theta^2 \), \( \theta \in R \), then we have

\begin{align*}
L(\cos^2 \theta) & = \frac{\cos^2 \theta}{3} f(a) g(a) + \frac{1 - \cos^2 \theta}{3} f(b) g(b) \\
& + \frac{1}{3} f(b \cos^2 \theta + a \sin^2 \theta) g(b \cos^2 \theta + a \sin^2 \theta) \\
& + \frac{1}{6} f(b \cos^2 \theta + a \sin^2 \theta) [g(a) \cos^2 \theta + g(b) \sin^2 \theta] \\
& + \frac{1}{6} g(b \cos^2 \theta + a \sin^2 \theta) [g(b) \cos^2 \theta + g(a) \sin^2 \theta]
\end{align*}