

A new Hermite-Hadamard Type Inequality

Xin Chen, Yaqi Chen* and Xiang Gao

School of Mathematical Sciences, Ocean University of China, Lane238, Songling Road, Laoshan District, Qingdao City, Shandong Province, 266100, People's Republic of China

*Corresponding author

Abstract—In this paper, a better estimate of the Hermite type inequality for the product of two convex functions is established, and the proof is given. Then the inferences and applications of the inequalities obtained are provided.

Keywords—Hermite-Hadamard integral inequality; Convex functions

I. INTRODUCTION

The following definition is well-known in the literature. A real-valued function $f : [a, b] \rightarrow \mathbb{R}$, $\emptyset \neq I = [a, b] \subseteq \mathbb{R}$, is said to be convex on $[a, b]$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Conversely, if the opposite inequality holds, the function is said to be concave on I . A function f is convex on I if and only if $f''(x) \geq 0$ for all $x \in (a, b)$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex mapping. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard's inequality for convex mapping.

So far, a lot of literatures have been researched on the famous Hermite-Hadamard inequality which was first published in [1]. In recent years, some scholars have tried to improve and generalize the classical inequality. For example, ABDALLAH EL FARISSI had given new estimates in [2]; In [3], Hadamard inequality is concerned with the compound of two convex functions, which is proposed by Xiang Gao.

II. MAIN RESULTS

In this part, we will establish a new estimate of the product of two convex functions, which is a generalization of the following inequality:

THEOREM 1.[3]

Let $f, g : [a, b] \rightarrow [0, \infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, $a < b$. Then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b) \quad (1)$$

and

$$\begin{aligned} & 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b) \end{aligned} \quad (2)$$

where

$$\begin{aligned} M(a, b) &= f(a)g(a) + f(b)g(b) \\ N(a, b) &= f(a)g(b) + f(b)g(a). \end{aligned}$$

According to the Theorem1, we can easily gain the following results.

COROLLARY 1.

Let $f, g : [a, b] \rightarrow [0, +\infty)$ be convex functions on $[a, b] \subset \mathbb{R}$. And then meet $[f(a) - f(b)][g(a) - g(b)] \leq 0$, we can learn that the product of f and g is still satisfied (1) and (2).

Secondly, we get the following main result.

(MAIN RESULT) THEOREM 2.

Let $f, g : [a, b] \rightarrow [0, +\infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, and meet $[f(a) - f(b)][g(a) - g(b)] \leq 0$. Then for all $\lambda \in [0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq L(\lambda) \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b) \quad (3)$$

and

$$\begin{aligned} & 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a, b) - \frac{1}{3}N(a, b) \\ & \leq l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx \end{aligned} \quad (4)$$

where

$$L(\lambda) := \frac{\lambda}{3} f(a)g(a) + \frac{1-\lambda}{3} f(b)g(b) \\ + \frac{1}{3} f(\lambda b + (1-\lambda)a)g(\lambda b + (1-\lambda)a) \\ + \frac{1}{6} f(\lambda b + (1-\lambda)a)[\lambda g(a) + (1-\lambda)g(b)] \\ + \frac{1}{6} g(\lambda b + (1-\lambda)a)[\lambda g(b) + (1-\lambda)g(a)]$$

and

$$l(\lambda) := 2\lambda f\left(\frac{(2-\lambda)a + \lambda b}{2}\right)g\left(\frac{(2-\lambda)a + \lambda b}{2}\right) \\ - \frac{1+3\lambda-3\lambda^2}{6} M(a,b) - \frac{2+3\lambda^2-3\lambda}{6} N(a,b) \\ + 2(1-\lambda)f\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)g\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)$$

COROLLARY 2.

If replace “ $[f(a) - f(b)][g(a) - g(b)] \leq 0$ ” with neither f nor g are increasing or decreasing synchronously. Then for all $\lambda \in [0,1]$, the results of Theorem 2 are still satisfied.

COROLLARY 3.

Let $f, g : [a,b] \rightarrow [0,\infty)$ be convex functions on $[a,b] \subseteq \mathbb{R}$, $a < b$, and meet $[f(a) - f(b)][g(a) - g(b)] \leq 0$.

Then for all $\lambda \in [0,1]$, we have the following inequality

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \inf_{\lambda \in [0,1]} L(\lambda) \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6} M(a,b) - \frac{1}{3} N(a,b) \\ \leq \sup_{\lambda \in [0,1]} l(\lambda) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx$$

where $l(\lambda)$, $L(\lambda)$ are defined in Theorem 2.

REMARK 1. Applying Theorem 2 for $\lambda=1$, we get inequality (1) and (2). In addition, if we choose $g(x) = 1$, then we have the right side of Hermite-Hadamard inequality.

III. LEMMA

In order to prove Theorem 2, we shall need the following Lemma:

LEMMA 1.

Let F be convex functions on $[a,b] \subseteq \mathbb{R}$, $a < b$, Then for all

$\lambda \in [0,1]$, we have

$$F\left(\frac{a+b}{2}\right) \leq \lambda F\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)F\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \quad (5)$$

Proof. Let

$$\frac{a+b}{2} = \lambda \frac{\lambda b + (2-\lambda)a}{2} + (1-\lambda) \frac{(1+\lambda)b + (1-\lambda)a}{2}$$

Then, applying the properties of the convex function, (5) can be proved easily. W

IV. PROOF OF THE THEOREMS

Proof of Theorem2.

Firstly, discuss about (3): Applying (1) on the subinterval

$[a, \lambda b + (1-\lambda)a]$, with $\lambda \neq 0$, we have

$$\frac{1}{\lambda(b-a)} \int_a^{\lambda b + (1-\lambda)a} f(x)g(x)dx \\ \leq \frac{1}{3} [f(a)g(a) + f(\lambda b + (1-\lambda)a)g(\lambda b + (1-\lambda)a)] \\ + \frac{1}{6} [f(a)g(\lambda b + (1-\lambda)a) + f(\lambda b + (1-\lambda)a)g(a)] \quad (6)$$

Applying (1) on the subinterval $[\lambda b + (1-\lambda)a, b]$, with $\lambda \neq 1$, we have

$$\frac{1}{(1-\lambda)(b-a)} \int_{\lambda b + (1-\lambda)a}^b f(x)g(x)dx \\ \leq \frac{1}{3} [f(\lambda b + (1-\lambda)a)g(\lambda b + (1-\lambda)a) + f(b)g(b)] \\ + \frac{1}{6} [f(\lambda b + (1-\lambda)a)g(b) + f(b)g(\lambda b + (1-\lambda)a)] \quad (7)$$

Multiplying (6) by λ , (7) by $(1-\lambda)$, then adding the inequalities, we get:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \\ \leq \frac{\lambda}{3} f(a)g(a) + \frac{1-\lambda}{3} f(b)g(b) \\ + \frac{1}{3} [\lambda f(b) + (1-\lambda)f(a)][\lambda g(b) + (1-\lambda)g(a)] \\ + \frac{1}{6} [\lambda f(b) + (1-\lambda)f(a)][\lambda g(a) + (1-\lambda)g(b)] \\ + \frac{1}{6} [\lambda g(b) + (1-\lambda)g(a)][\lambda g(b) + (1-\lambda)g(a)] \\ = \frac{1}{3} [f(a)g(a) + f(b)g(b)] + \frac{1}{6} [f(b)g(a) + f(a)g(b)]$$

$$= \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)$$

So, we get that $L(\lambda)$ is defined as in Theorem 2.

Secondly, discuss about (4):

According to (4), it also can be written as

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \geq 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a, b) - \frac{1}{3}N(a, b)$$

Using the same method of above, we get

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \geq 2\lambda f\left(\frac{(2-\lambda)a + \lambda b}{2}\right)g\left(\frac{(2-\lambda)a + \lambda b}{2}\right) \\ & \quad - \frac{\lambda}{6} [f(a)g(a) + f(\lambda b + (1-\lambda)a)g(\lambda b + (1-\lambda)a)] \\ & \quad - \frac{\lambda}{3} [f(a)g(\lambda b + (1-\lambda)a) + f(\lambda b + (1-\lambda)a)g(a)] \\ & \quad + 2(1-\lambda)f\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)g\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right) \\ & \quad - \frac{1-\lambda}{6} [f(\lambda b + (1-\lambda)a)g(\lambda b + (1-\lambda)a) + f(b)g(b)] \\ & \quad - \frac{1-\lambda}{3} [f(\lambda b + (1-\lambda)a)g(b) + f(b)g(\lambda b + (1-\lambda)a)] \\ & \geq 2\lambda f\left(\frac{(2-\lambda)a + \lambda b}{2}\right)g\left(\frac{(2-\lambda)a + \lambda b}{2}\right) \\ & \quad + 2(1-\lambda)f\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)g\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right) \quad (8) \\ & \quad - \frac{1+3\lambda-3\lambda^2}{6} M(a, b) - \frac{2+3\lambda^2-3\lambda}{6} N(a, b) \end{aligned}$$

Let $F(x) = f(x)g(x)$, according to Lemma 1 we get

$$\begin{aligned} (8) & \geq 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \quad - \frac{1+3\lambda-3\lambda^2}{6} M(a, b) - \frac{2+3\lambda^2-3\lambda}{6} N(a, b) \quad (9) \end{aligned}$$

Because of $[f(a) - f(b)] \cdot [g(a) - g(b)] \leq 0$,

$$(9) \geq 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a, b) - \frac{1}{3}N(a, b)$$

Then, we get $l(\lambda) \cdot W$

EXAMPLE .

Let $\lambda = \cos^2 \theta$, $\theta \in R$, then we have

$$\begin{aligned} L(\cos^2 \theta) &= \frac{\cos^2 \theta}{3} f(a)g(a) + \frac{1-\cos^2 \theta}{3} f(b)g(b) \\ &+ \frac{1}{3} f(b \cos^2 \theta + a \sin^2 \theta)g(b \cos^2 \theta + a \sin^2 \theta) \\ &+ \frac{1}{6} f(b \cos^2 \theta + a \sin^2 \theta)[g(a) \cos^2 \theta + g(b) \sin^2 \theta] \\ &+ \frac{1}{6} g(b \cos^2 \theta + a \sin^2 \theta)[g(b) \cos^2 \theta + g(a) \sin^2 \theta] \end{aligned}$$

and

$$\begin{aligned} l(\cos^2 \theta) &= 2 \cos^2 \theta f\left(\frac{a(1+\sin^2 \theta) + b \cos^2 \theta}{2}\right)g\left(\frac{a(1+\sin^2 \theta) + b \cos^2 \theta}{2}\right) \\ &+ 2 \sin^2 \theta f\left(\frac{a \sin^2 \theta + b(1+\cos^2 \theta)}{2}\right)g\left(\frac{a \sin^2 \theta + b(1+\cos^2 \theta)}{2}\right) \\ &- \frac{1+3 \cos^2 \theta - 3 \cos^4 \theta}{6} M(a, b) - \frac{2-3 \cos^2 \theta + 3 \cos^4 \theta}{6} N(a, b) \end{aligned}$$

ACKNOWLEDGMENTS

I would especially like to express my appreciation to my advisor professor Xiang Gao for longtime encouragement and meaningful discussions. I would also especially like to thank the referee for meaningful suggestions that led to improvement of the article.

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