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The Hyperbolic Method Algorithm for the Optimal Determination of Coordinates and the Evaluation of the Potential Accuracy with Which They Are Measured

Vladimir P. Dubykin[1], Konstantin S. Kalashnikov[2,*], Boris V. Matveev[1] and Oleg V. Chernoyarov[3, 4]
[1]Dept. of Radio Engineering, Voronezh State Technical University, Moscow Avenue, 14, Voronezh, 394026, Russia
[2]Dept. of Radio Physics, Voronezh State University, Universitetskaya sq., Voronezh, 394018, Russia
[3]Dept. of Electronics and Nanoelectronics, National Research Institute "Moscow Power Engineering Institute", Krasnokazarmennaya st, 14, Moscow, 111250, Russia
[4]International Laboratory of Statistics of Stochastic Processes and Quantitative Finance, National Research Tomsk State University, Lenin Avenue 36, Tomsk, 634050, Russia
[*]Corresponding author

Abstract—**We present the analytical relations and the order of their introduction in the hyperbolic method algorithm designed for the generation of the optimal estimates of the radio sources coordinates and for the evaluation of the potential accuracy with which they are measured in the time-difference of arrival system within the Cartesian coordinate system.**

Keywords—TDOA system; radio source; coordinate measurement errors; maximum likelihood; simulation

I. Introduction

At present, great attention is paid to radio sources (RS) coordinate measuring by the multiposition passive radar systems [1-4], due to the functional features of such radar systems described in detail in, for example, [2, 4]. In these papers, the following methods for the determination of RS coordinates are considered: elliptic, hyperbolic, triangulation, and also their combinations. It was shown that if the time of propagation of radio signal from RS to the each of the receiving stations of the system cannot be measured and it is not required to know a priori the form and the emission time of the received signals, then the hyperbolic method is considered as more preferable for the determination of the RS coordinates (including active interference sources). This method is based on the compiling of nonlinear equations derived from the time-difference of arrival (TDOA) system geometry. However, the common methods of dealing with that kind of combined equations do not provide the maximum likelihood estimates of the RS coordinates [5].

Thus, the purpose of the present paper is to develop the optimal (maximum likelihood) hyperbolic method algorithm for the determination of the RS coordinates and for the evaluation of the potential accuracy with which they are measured. In this case, hyperbolic method is modified as it differs from the common one by the links revealed between RS coordinates and the statistical properties of the random errors characteristics for the measurement of differences in distances between RS and each of the receiving stations. These links are specified through the distribution law of the reciprocal delays of RS signal propagation to the two neighboring receiving stations of TDOA system.

II. TDOA System Singularities and Operating Conditions

In the design of the proposed algorithm, the following TDOA system features and operating conditions are taken into account:

1. As we deal with in-plane RS position determination, the geometric structure of TDOA system represents an arbitrary triangle where the receiving stations are placed in vertexes of triangle and are set in the Cartesian right-handed coordinate system. Our designations are: (x_i, y_i) – the coordinates of the i-th receiving station ($i=\overline{0,M-1}$), B_{im} – the distance between two neighboring receiving stations (base), M – the number of receiving stations, and (x, y) – the current RS coordinates. Due to the closed geometric structure of TDOA system, the numeration of the receiving stations can begin with any receiving station and continue along a structure contour in a counterclockwise direction. Then, the base designation is carried out by the two indexes – i and m, where $m=i+1$, if $i=0,1,\dots M-2$, and $m=0$, if $i=M-1$. In particular, for the three-position TDOA system considered below there are three bases with the designations B_{01}, B_{12}, B_{20}. Since the initial RS signal emission time is unknown, the relative times of arrival (reciprocal delays) τ_{im} of the RS signal to the each of the two neighboring receiving stations of TDOA system are measured.

2. The RS coordinates measurement errors obey the normal law. In addition, the measurements τ_{im} are independent for

each base B_{im}, while the estimate of the RS coordinates measurement accuracy is the maximum likelihood one [4].

3. The proposed algorithm operates at the final stage of the two-phase algorithm, the initial data for which are the mutual delays τ_{im}.

4. In order to obtain the maximum likelihood estimate of RS coordinates and the correlation matrix of errors of their measurements, the iterative method [1-4] is applied when at each n-th step ($n = 1,2,\dots$) of the iterative process the conditional probability density of coordinates $w(x, y|x_{n-1}, y_{n-1})$ is formed, provided that at the previous step the coordinates have possessed the values x_{n-1}, y_{n-1}. Iterative process is terminated when the square of the distance between the points (x_n, y_n) and (x_{n-1}, y_{n-1}) does not exceed the agreed threshold value. And for choosing the initial point of the iterative process with the coordinates (x_0, y_0) the following methods [6] are applied: the method of TDOA system geometric center, the method of weighed (concerning received RS signal power) TDOA system geometric center, and the method of intersection of asymptotes to hyperbola of all the pairs of TDOA system bases B_{im}. The optimal estimate of RS coordinates is made under the maximum value of the probability density of coordinates [4, 7], i.e.

$$\max w(x_n, y_n|x_{n-1}, y_{n-1}) = w(\hat{x}_n, \hat{y}_n|x_{n-1}, y_{n-1}). \qquad (1)$$

III. The Procedure of the Optimal Estimate of RS Coordinates

In view of the foregoing features and in order to obtain the coordinate relations between the known coordinates (X_i, Y_i) of the receiving stations of TDOA system and the RS coordinates (x, y) to be estimated, we write down the joint probability density of the mutual delays measurement errors as

$$w(\delta\tau_{01}, \delta\tau_{12}, \delta\tau_{20}) = \prod_{(i,m)} w(\delta\tau_{im}). \qquad (2)$$

Here $\delta\tau_{im} = \tau_{im} - \tau_{im}^{(0)}$, τ_{im}, $\tau_{im}^{(0)}$ are the measured and true mutual delays, respectively,

$$w(\delta\tau_{im}) = \exp\left[-\left(\tau_{im} - \tau_{im}^{(0)}\right)^2 \Big/ 2\sigma_{\tau_{im}}^2\right] \Big/ \sigma_{\tau_{im}}\sqrt{2\pi}, \qquad (3)$$

and $\sigma_{\tau_{im}}$ is the root-mean-square deviation of the mutual delays measurement.

Further, we express the errors $\delta\tau_{im}$ through possible RS coordinates (x, y) by the obvious equation

$$\delta\tau_{im} = [D_i(x, y) - D_m(x, y)]/c - \tau_{im}. \qquad (4)$$

Here $D_l(x, y) = \sqrt{(x - X_l)^2 + (y - Y_l)^2}$, $l = i, m$ is the distance from the RS with unknown coordinates (x, y) to the l-th receiving station of TDOA system, c is a light speed.

By substituting Eqs. (3), (4) into Eq. (2) in order to find the expression for probability density $w(x, y|x_{n-1}, y_{n-1})$, we obtain the complex nonlinear equation. Then, we overcome the difficulties accompanying its solution by applying the linearization technique [2, 3]. For this purpose, we introduce designation $\Gamma_{im} = D_i(x, y) - D_m(x, y)$ and linearize the function Γ_{im} by expanding it into a Taylor series in the neighborhood of the point with coordinates (x_{n-1}, y_{n-1}), being abridged to the linear expansion terms. Then we have

$$\Gamma_{im}(x, y) = \Gamma_{im}(x_{n-1}, y_{n-1}) + A_{im}(x - x_{n-1}) + B_{im}(y - y_{n-1}). \qquad (5)$$

where $A_{im} = \partial\Gamma_{im}(x_{n-1}, y_{n-1})/\partial x$, $B_{im} = \partial\Gamma_{im}(x_{n-1}, y_{n-1})/\partial y$.

We replace the expressions A_{im} and B_{im} by the equations with linear differences according to the Lagrang's theorem [8] and get

$$\begin{aligned} A_{im} &= \frac{x_{n-1} - X_i}{D_i(x_{n-1}, y_{n-1})} - \frac{x_{n-1} - X_m}{D_m(x_{n-1}, y_{n-1})}, \\ B_{im} &= \frac{y_{n-1} - Y_i}{D_i(x_{n-1}, y_{n-1})} - \frac{y_{n-1} - Y_m}{D_m(x_{n-1}, y_{n-1})}. \end{aligned} \qquad (6)$$

By substituting Eqs. (3)-(6) into Eq. (2), after rather simple mathematical manipulations, we obtain the desired probability distribution law for the coordinates (x, y) in the form of

$$w(x, y|x_{n-1}, y_{n-1}) = C_1 \exp[-\lambda(x, y)/2], \qquad (7)$$

where C_1 is the multiplicative constant and

$$\lambda(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F. \qquad (8)$$

In Eq. (8), the designations are

$$A = \sum_{i,m} \frac{A_{im}^2}{\sigma_{\Gamma_{im}}^2}, \quad B = \sum_{i,m} \frac{A_{im}B_{im}}{\sigma_{\Gamma_{im}}^2}, \quad C = \sum_{i,m} \frac{B_{im}^2}{\sigma_{\Gamma_{im}}^2},$$

$$D = \sum_{i,m} \frac{A_{im}e_{im}}{\sigma_{\Gamma_{im}}^2}, \quad E = \sum_{i,m} \frac{B_{im}e_{im}}{\sigma_{\Gamma_{im}}^2}, \quad F = \sum_{i,m} e_{im}^2,$$

$$e_{im} = \Gamma_{im} - x_{n-1}A_{im} - y_{n-1}B_{im} - c\tau_{im}, \qquad \sigma^2_{\Gamma_{im}} = c^2\sigma^2_{\tau_{im}}.$$

The optimal estimate of the coordinates (x, y) by the maximum likelihood method is determined from Eq. (1). Then we maximize the expression (7) in a certain way [2, 3, 7] and obtain the following equations

$$\partial\lambda(x, y)/\partial x\big|_{\substack{x=\hat{x}_n\\ y=\hat{y}_n}} = A\hat{x}_n + B\hat{y}_n + D = 0,$$

$$\partial\lambda(x, y)/\partial y\big|_{\substack{x=\hat{x}_n\\ y=\hat{y}_n}} = B\hat{x}_n + C\hat{y}_n + E = 0.$$

From here we get

$$\hat{x}_n = \frac{BE - CD}{AC - B^2}, \qquad \hat{y}_n = \frac{BD - AE}{AC - B^2}. \qquad (9)$$

IV. THE COORDINATE MEASURING ACCURACY EVALUATION

From the statistical estimation theory, it is well known that when RS signal observation time increases or/and RS signal significantly exceeds the noise level at the receiver output, the maximum likelihood estimates are asymptotically unbiased, effective, asymptotically normal and their correlation matrix is inverse to the information Fisher matrix [4, 7]. In relation to the estimates (9), it can be easy demonstrated that the correlation matrix K and the matrix K^{-1} inverse to it are determined as

$$K = \begin{pmatrix} C/\Delta & -B/\Delta \\ -B/\Delta & A/\Delta \end{pmatrix}, \qquad K^{-1} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}. \qquad (10)$$

where $\Delta = AC - B^2 \neq 0$ is the determinant of matrix K^{-1}.

On the other hand, the matrix K can be written down as [2-4]

$$K = \begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_y r_{xy} \\ \sigma_x\sigma_y r_{xy} & \sigma_x^2 \end{pmatrix}, \qquad (11)$$

where σ_x, σ_y and r_{xy} are roof-mean-square deviations and correlation coefficient of the coordinates estimates, correspondingly. Thus, having compared how Eqs. (10) and (11) present the RS coordinates measurement errors, we get

$$\sigma_x^2 = C/\Delta, \qquad \sigma_y^2 = A/\Delta, \qquad r_{xy} = -B/AC. \qquad (12)$$

V. THE CALCULATIONS SEQUENCE

For the introduced algorithm, the path for the determination of RS coordinates, as well as the calculation of accuracy of their measurement, may be summarized as follows.

1. In rectangular (Cartesian) right-handed coordinate system the known coordinates of the receiving stations of TDOA system are set and the unknown RS coordinates are introduced.

2. By applying the relation (4) and the chosen TDOA system geometry, the differences of distances Γ_{im} are determined between RS and receiving stations with the base B_{im}.

3. In accordance with Eqs. (5) and (6), the functions $\Delta D_{im}(x, y)$ are expanded into Taylor series by the first power expansion.

4. The conditional probability density function (7) of coordinates (x, y) and its quadratic form (8) are generated.

5. The function (7) is maximized and the optimal RS coordinates are calculated by the formulas (9).

6. The accuracy of the measurement of the found coordinates is estimated through Eqs. (10)-(12).

VI. THE ALGORITHM OPERATION TEST

To confirm the operability both of the introduced technique and the algorithm synthesized on its basis, the mathematical simulation is carried out of the operation of the three-position TDOA system (using MATLAB application package). The receiving stations of TDOA system are situated in vertexes of equilateral triangle with the centre at the coordinate origin and the base length of $B_{im} \approx 10$ km. The pass-band of TDOA system receivers is 20 MHz. The RS signals bandwidth varies between 20 and 200 kHz, while their power is 10 W.

As a test signal, the chirp signal is used as the simplest generated one that allows controlling bandwidth of the signals from various RSs. The mathematical model of the test signal at the input of m-th receiving station of TDOA system is produced in the form of

$$S_m(t) = \sum_{l=1}^{I}\sqrt{\frac{P_{cl}}{4\pi D_{ml}^2}}\cos\left[2\pi f_{cl}(t-\tau_{ml}) + \frac{\pi\Delta f_{cl}}{T_{cl}}(t-\tau_{ml})^2\right] + \frac{\xi_{gauss}(t)}{q_m}\sqrt{\sum_{l=1}^{L}\frac{P_{cl}}{4\pi D_{ml}}}.$$

Here L is the number of RSs; P_{cl}, T_{cl}, f_{cl}, Δf_{cl} are the power, duration, carrier frequency and bandwidth of the signal from l-th RS, correspondingly, D_{ml} and τ_{ml} are the distance and the signal propagation delay from l-th RS to m-th receiving station; $\xi_{gauss}(t)$ is the Gaussian random process with zero mathematical expectation, unit variance and uniform spectral density within receiving station pass-band; q_m^2 is the power signal-to-noise ratio at the *m*-th receiving station input.

Some simulation results for various methods of the reference point (x_0, y_0) choice are shown in Table I. The values provided here are obtained by averaging over 100 measurements of RS coordinates under the fixed duration of each measurement and under the fixed coordinate and signal conditions.

TABLE I. THE SIMULATION RESULTS FOR VARIOUS METHODS OF THE REFERENCE POINT SELECTION

Distance, km	Reference Point Selection Method					
	Geometrical center		***Weighted center***		***Crossings of hyperbolas asymptotics***	
	root-mean-square error, m	***Number of iteration***	***root-mean-square error, m***	***Number of iteration***	***root-mean-square error, m***	***Number of iteration***
5	2.72	3	1.76	3	0.32	3
8	2.13	4	1.43	3	3.53	4
10	2.17	7	1.48	4	4.30	12
12	3.0	9	2.43	9	5.74	16
20	4.63	13	4.84	17	3.47	11

The analysis of simulation results, as a whole, shows that the developed algorithm is operable. It allows us to estimate TDOA system operating efficiency in a wide range of conditions, in different coordinate and interference settings. The additional advantage of the presented algorithm is that its application makes it possible to estimate the operational environment when TDOA system is working with several RSs simultaneously and when, depending on the tasks to be solved, we use different methods of the reference point selection for the iterative procedure.

VII. CONCLUSION

We introduce the hyperbolic method algorithm for the optimal determination of coordinates and the evaluation of the potential accuracy of their measurement. The determination of optimal RS coordinates and the estimation of their accuracy are carried out through the conditional posterior probability density of coordinates, with the statistical estimation methods and the Fisher matrix applied. The algorithm realizes the basic analytical relations in an explicit form that allows developers of measurement procedures to promptly generate various preliminary expert estimates at the different measurement process stages.

As the initial data for the algorithm, the mutual delays of signals from radiating objects are used, the said signals propagating to each pair of the receiving positions, the optimal measurement estimates of which are the maximum likelihood ones.

We implement the optimal estimation of RS coordinates and propose the sequence of their determination, together with the estimation of the measurement accuracy. The algorithm structure allows us to carry out, if necessary, interfilter processing of the signals for their identification relative to radio sources emitting them. The simulation of the algorithm has proved its operability for the three-position TDOA system.

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