

An Elementary Proof for a Class of Integral Inequality System

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Abstract: The purpose of this paper is to establish some results for the class of system of integral inequalities by means of variable transformation method and Gronwall-Bellman's inequality. These results presented in this paper generalize and improve some recent results. Meanwhile, two examples indicate the effectiveness of the main results.

1. Introduction

It is well known that the differential and integral inequalities plays an important role in the study of quantitative properties and stability of solutions of differential and integral equations (see, e.g., [2, 3, 5, 6] and references therein). The celebrated Gronwall's inequality which was established in 1919 by Gronwall^[4] occurs very frequently in the theory of differential equations in various context. According to the different motivations, various inequalities and their generalized forms of Gronwall's inequality have been obtained and applied extensively in diversity areas including global existence, uniqueness, stability, boundary value problem, and other properties (see, e.g., [1–7] and references therein). In particular, Bellman [1] gave the Gronwall-Bellman's inequality as following:

Lemma 1^[1, 6] (Gronwall-Bellman's inequality) suppose $\phi(t)$ and $b(t)$ are nonnegative continuous functions on $J = [\alpha, \beta]$. Assume that $\phi(t) \leq a + \int_{\alpha}^t b(s)\phi(s)ds, \forall t \in J$, where $a \geq 0$ is a constant. Then $\phi(t) \leq ae^{\int_{\alpha}^t b(s)ds}, \forall t \in J$.

In 1963, Bellman and Cook [2] generalized the above inequality as following:

Lemma 2^[2, 5] (Generalized Gronwall-Bellman's inequality) suppose $\phi(t)$ and $b(t)$ are nonnegative continuous functions on $J = [0, T)(T \leq +\infty)$. Assume that $c(t) > 0$ is a monotone nondecreasing function on J . If $\phi(t) \leq c(t) + \int_{\alpha}^t b(s)\phi(s)ds, \forall t \in J$ holds, then $\phi(t) \leq c(t)e^{\int_{\alpha}^t b(s)ds}, \forall t \in J$.

The purpose of this paper is to prove a class of integral inequality system by the generalized

Gronwall-Bellman's inequality and variable transformation method. Meanwhile, we shall give two examples as an application. These results presented in this paper generalize and improve some recent results of Xuerong Mao in 1989^[6].

2. The Main results

One main results are given in the following the theorems:

Theorem 1. Suppose $u(t), v(t)$ and $h_i(t) (i=1,2,3,4)$ are nonnegative continuous functions on $[0, \infty)$, and let $c_j(t) (j=1,2)$ be positive and monotone continuous function on $[0, \infty)$. Assume that there exists a nonnegative continuous function $\theta(t)$ such that $h_i(t) \leq \theta(t) (i=1,2,3,4)$ for any $t > 0$. If the following inequalities (1)

$$u(t) \leq c_1(t) + \int_0^t h_1(s)v(s)e^{\mu s} ds + \int_0^t h_3(s)u(s)ds, v(t) \leq c_2(t) + \int_0^t h_2(s)u(s)e^{-\mu s} ds + \int_0^t h_4(s)v(s)ds$$

hold on $[0, \infty)$, where $\mu \geq 0$ is a constant. Then there exist continuous functions $C(t) > 0$ and $\rho_k(t) \geq 0 (k=1,2)$ on $[0, \infty)$ such that $u(t) \leq C(t)e^{\int_0^t \rho_1(s)ds}$, $v(t) \leq C(t)e^{\int_0^t \rho_2(s)ds}$.

Proof First of all, let us make the transformation

$$u(t) = \tilde{u}(t)e^{-\alpha t}, v(t) = \tilde{v}(t)e^{-\beta t}, t \in [0, \infty), \quad (2)$$

where α, β satisfy $\alpha + \mu = \beta, \alpha > 0$. Thus we substitute (2) into (1), and in view of $\alpha + \mu = \beta$, we obtain

$$\tilde{u}(t)e^{-\alpha t} \leq c_1(t) + \int_0^t (h_1(s)\tilde{v}(s) + h_3(s)\tilde{u}(s))e^{-\alpha s} ds, \tilde{v}(t)e^{-\beta t} \leq c_2(t) + \int_0^t (h_2(s)\tilde{u}(s) + h_4(s)\tilde{v}(s))e^{-\beta s} ds,$$

Meanwhile, since $h_i(t) \leq \theta(t) (i=1,2,3,4)$ on $(0, \infty)$, we get

$$\tilde{u}(t) \leq c_1(t)e^{\alpha t} + \int_0^t (\tilde{v}(s) + \tilde{u}(s))\theta(s)e^{\alpha(t-s)} ds, \tilde{v}(t) \leq c_2(t)e^{\beta t} + \int_0^t (\tilde{v}(s) + \tilde{u}(s))\theta(s)e^{\beta(t-s)} ds.$$

$$\begin{aligned} \text{Thus } \tilde{u}(t) + \tilde{v}(t) &\leq c_1(t)e^{\alpha t} + c_2(t)e^{\beta t} + \int_0^t (\tilde{u}(s) + \tilde{v}(s))\theta(s)(e^{\alpha(t-s)} + e^{\beta(t-s)}) ds \\ &\leq (c_1(t) + c_2(t))e^{\beta t} + \int_0^t 2\theta(s)(\tilde{u}(s) + \tilde{v}(s))e^{\beta(t-s)} ds, \forall t \geq 0. \end{aligned}$$

$$\text{Namely } (\tilde{u}(t) + \tilde{v}(t))e^{-\beta t} \leq (c_1(t) + c_2(t)) + \int_0^t 2\theta(s)(\tilde{u}(s) + \tilde{v}(s))e^{-\beta s} ds, \forall t \geq 0.$$

It follows from Lemma 2 that

$$(\tilde{u}(t) + \tilde{v}(t))e^{-\beta t} \leq C(t)e^{\int_0^t 2\theta(s)ds}, \forall t \geq 0, \quad (3)$$

where $C(t) = c_1(t) + c_2(t)$. Setting $\rho_1(t) = 2\theta(t) + \mu t, \rho_2(t) = 2\theta(t)$, it follows from (2) and (3) that we obtain the conclusion of the Theorem 1. This completes the proof.

Example 1 Consider the following initial-value problem

$$\begin{cases} u'(t) \leq 4t + 1 + 2^{-t}v(t)e^{2t} + (\sqrt{t} + 1)u(t), \forall t \in [0, +\infty), \\ v'(t) \leq 2^t \ln 2 + (1 + t^2)^{-1}u(t)e^{-2t} + (1 + 3^{-1}t)v(t), \forall t \in [0, +\infty), \\ u(0) = 1, v(0) = \ln 2. \end{cases} \quad (4)$$

where $h_1(t) = 2^{-t}, h_2(t) = (1 + t^2)^{-1}, h_3(t) = \sqrt{t} + 1, h_4(t) = 1 + 3^{-1}t$. Set $\theta(t) = \sqrt{t} + 2$. Then we know that $0 \leq h_i(t) \leq \theta(t) (i=1,2,3,4)$ for $\forall t \in [0, \infty)$. Hence, by a direct calculation and using Theorem 1, it can be easily shown that the solutions $(u(t), v(t))$ of system (4) have a pair of priori upper control functions respectively in the following form: $u(t) \leq (2t^2 + t + 2^t)e^{\frac{4}{3}\sqrt{3t^3} + 2t + t^2}$, $v(t) \leq (2t^2 + t + 2^t)e^{\frac{4}{3}\sqrt{3t^3} + 2t}, \forall t \in [0, +\infty)$.

If $h_3(t) \equiv 0 \equiv h_4(t)$ on $[0, \infty)$ in Theorem 1, then we have the following corollary:

Corollary 1 Let $u(t), v(t), h_i(t) (i=1,2,3,4)$ and $c_j(t) (j=1,2)$ be same as Theorem 1. If the following inequalities $u(t) \leq c_1(t) + \int_0^t h_1(s)v(s)e^{\mu s} ds, v(t) \leq c_2(t) + \int_0^t h_2(s)u(s)e^{-\mu s} ds$ hold on $[0, \infty)$, where $\mu \geq 0$ is a constant. Then there exist continuous functions $C(t) > 0$ and $\rho_k(t) \geq 0 (k=1,2)$ on $[0, \infty)$ such that $u(t) \leq C(t)e^{\int_0^t \rho_1(s)ds}, v(t) \leq C(t)e^{\int_0^t \rho_2(s)ds}$.

The following theorem shows some structure of solutions in the case when $c_i(t) \equiv c_i \geq 0 (i=1,2)$

on $[0, \infty)$ in Theorem 1.

Theorem 2 Let $u(t), v(t), h_i(t) (i=1,2,3,4)$ and $\theta(t)$ be same as Theorem 1. If the following inequalities

$$u(t) \leq c_1 + \int_0^t h_1(s)v(s)e^{\mu s} ds + \int_0^t h_3(s)u(s)ds, v(t) \leq c_2 + \int_0^t h_2(s)u(s)e^{-\mu s} ds + \int_0^t h_4(s)v(s)ds \quad \text{hold,}$$

where c_1, c_2, μ are nonnegative constants. Then there exist a constant $M > 0$ and continuous functions

$$\rho_k(t) \geq 0 (k=1,2) \text{ on } [0, \infty) \text{ such that } u(t) \leq Me^{\int_0^t \rho_1(s)ds}, v(t) \leq Me^{\int_0^t \rho_2(s)ds}.$$

Proof If we make the transform (2) and use Lemma 1, then the following proof is the same as Theorem 1, we omit the proof. This completes the proof.

Remark 1. If we set $\theta(t) \equiv M > 0$ on $[0, \infty)$ in Theorem 2, it is easy to see that our result extends the conclusion of Xuerong Mao in 1989 [6].

If $c_i(t) \equiv c_i \geq 0 (i=1,2)$ on $[0, \infty)$ in Corollary 1 or $h_3(t) \equiv h_4(t) \equiv 0$ on $[0, \infty)$ in Theorem 2, then we have the following corollary:

Corollary 2. Let $u(t), v(t), h_i(t) (i=1,2,3,4)$ and $\theta(t)$ be same as Theorem 1. If

$$u(t) \leq c_1 + \int_0^t h_1(s)v(s)e^{\mu s} ds, v(t) \leq c_2 + \int_0^t h_2(s)u(s)e^{-\mu s} ds$$

hold on $[0, \infty)$, where c_1, c_2, μ are nonnegative constant. Then there exist a constant $M > 0$ and nonnegative continuous functions $\rho_k(t) \geq 0 (k=1,2)$ on $[0, \infty)$, such that

$$u(t) \leq Me^{\int_0^t \rho_1(s)ds}, v(t) \leq Me^{\int_0^t \rho_2(s)ds}.$$

Example 2. Consider the following initial-value problem

$$\begin{cases} u'(t) \leq (3^{-2t} + 1)v(t)e^{4t}, \forall t \in [0, +\infty), \\ v'(t) \leq (1 + \sqrt{1 - 2^{-t}})u(t)e^{-4t}, \forall t \in [0, +\infty), \\ u(0) = 3, v(0) = 1. \end{cases} \quad (5)$$

It is easy to see that $1 \leq h_1(t) = 3^{-2t} + 1 \leq 2$ and $1 \leq h_2(t) = 1 + \sqrt{1 - 2^{-t}} \leq 2$ for $\forall t \in (0, +\infty)$. Let $\theta(t) = 2, \forall t \in [0, +\infty)$. By a direct calculation and using Corollary 2, we can easily show that the solutions $(u(t), v(t))$ of system (5) have a pair of priori upper control functions respectively in the following form: $u(t) \leq 4e^{4t+2t^2}, v(t) \leq 4e^{4t}, \forall t \in [0, +\infty)$.

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