

The Reciprocal Sums of the Pell Numbers

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Abstract—In this paper, we consider the reciprocal sums of the Pell numbers by elementary methods. First, we investigate the reciprocals sums of two products of Pell numbers. Then, we study the alternating reciprocals sums of two products of Pell numbers. Last, we consider the reciprocal sums of even and odd terms in the Pell sequence. We obtain some interesting identities involving the Pell numbers.

Keywords—pell numbers; reciprocal sum; identity

I. INTRODUCTION

F_n is defined by the second-order linear recurrence sequences

$$F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1, (1.1)$$

The Fibonacci numbers can be defined by Binet's formulae $F_n = \alpha^n - \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$, (1.2)

There are many interesting results on the properties of Fibonacci sequences; see [1–6]. Ohtsuka and Nakamura [1] studied the properties of the Fibonacci numbers and proved the following two interesting identities:

$$k = n \infty 1 F_k - 1 = F_{n-2}, \quad \text{if } n \text{ is even and } n \geq 2, F_{n-2} - 1, \text{ if } n \text{ is odd and } n \geq 1. \quad (1.3)$$

$$k = n \infty 1 F_k - 1 = F_n F_{n-1} - 1, \quad \text{if } n \text{ is even and } n \geq 2, F_n F_{n-1} - 1, \text{ if } n \text{ is odd and } n \geq 1. \quad (1.4)$$

x is the floor function; that is, it denotes the greatest integer less than or equal to x .

Wang AYZ, Zhang F. [3] researched the properties of the Fibonacci numbers and proved the following four interesting identities:

$$k = nmn1 F_2 k - 1 = F_{2n-1}, \quad \text{if } m=2 \text{ and } n \geq 3, F_{2n-1} - 1, \text{ if } m \geq 3 \text{ and } n \geq 1. \quad (1.5)$$

$$k = nmn1 F_2 k - 1 = F_{2n-2}, \text{ For all } n \geq 1 \text{ and } m \geq 2. \quad (1.6)$$

$$k = nmn1 F_2 k - 1 = F_{4n-2}, \text{ For all } n \geq 1 \text{ and } m \geq 2. \quad (1.7)$$

$$k = nmn1 F_2 k - 1 = F_{4n-4}, \text{ For all } n \geq 1 \text{ and } m \geq 2. \quad (1.8)$$

Wang AYZ, Liu RN [4] demonstrated the following identities for the Fibonacci numbers

$$k = nmn1 F_k F_k - 1 = F_{n2}, \quad \text{if } n \text{ is even, } F_{n2} - 1, \text{ if } n \text{ is odd.} \quad (1.9)$$

$$k = nmn1 F_2 k F_2 k - 1 = F_{4n-3}, \text{ For all } n \geq 2 \text{ and } m \geq 2. \quad (1.10)$$

P_n is also defined by the second-order are

$$P_{n+2} = 2P_{n+1} + P_n, P_0 = 0, P_1 = 1. \quad (1.11)$$

The Pell numbers also provide boundless opportunities to experiment, explore, and conjecture, they are a lot of fun for inquisitive amateurs and professionals alike. The authors [7] and [8] studied the infinite sums derived from the Pell numbers and proved the following identities:

$$k = n \infty 1 P_k - 1 = P_{n-1} + P_{n-2}, \quad \text{if } n \text{ is even and } n \geq 2, P_{n-1} + P_{n-2} - 1, \text{ if } n \text{ is odd and } n \geq 1. \quad (1.12)$$

$$k = n \infty 1 P_k - 1 = 2P_{n-1} + P_{n-1}, \quad \text{if } n \text{ is even and } n \geq 2, 2P_{n-1} + P_{n-1}, \text{ if } n \text{ is odd and } n \geq 1. \quad (1.13)$$

Xu and Wang [9] proofed the following interesting identities for the Pell numbers:

$$k = n \infty 1 P_k - 1 = P_n 2P_{n-1} + 3P_n P_{n-2} + \dots - 6182P_n - 9182P_{n-1}, \quad \text{if } n \text{ is even and } n \geq 2, P_n 2P_{n-1} + 3P_n P_{n-2} + \dots - 6182P_n - 9182P_{n-1}, \text{ if } n \text{ is odd and } n \geq 1. \quad (1.14)$$

Applying elementary methods, we investigate the partial finite sums of the Pell numbers in this paper, and obtain some interesting families of identities. In section 2, we consider the sums of products of two reciprocals. In section 3, we study the alternating sums of products of two reciprocals. In section 4, we consider the reciprocal sums of even and odd terms in the Pell sequence.

II. RESULTS I: THE RECIPROCAL SUMS

$x^2 - 2x - 1 = 0$, the Pell numbers can be defined by Binet-like formulae:

$$P_n = \gamma^n - \delta^n \gamma - \delta, \text{ where } \gamma = 1 + 2, \delta = 1 - 2.$$

Using the Binet-like formulae, we can obtain properties of Pell numbers:

Lemma 2.1 Let m, n, s, t be positive integers

$$P_m P_n - P_s P_t = (-1)^{m+1} P_{n-s} P_{n-t}, (n > \max\{s, t\}, m + n = s + t) \quad (2.1)$$

Lemma 2.2

$$P_m P_n + P_{m+1} P_{n+1} = P_{m+n+1}. \quad (2.2)$$

As consequence of (2.2), we have the following result:

Corollary 2.3

$$P_n^2 + P_{n+1}^2 = P_{2n+1}. \quad (2.3)$$

Corollary 2.4

$$P_{n-1}P_{n+1} + P_nP_{n+2} = P_{2n+1}. \quad (2.4)$$

Lemma 2.5

$$P_{n+1}P_{n+2} - P_{n-1}P_n = 2P_{2n+1}. \quad (2.5)$$

Proof Applying (2.4) and recursion formula of Pell numbers

$$\begin{aligned} 2P_{2n+1} &= 2P_{n-1}P_{n+1} + 2P_nP_{n+2} \\ &= (P_{n+2} - P_n)P_{n-1} + (P_{n+1} - P_{n-1})P_{n+2} \\ &= P_{n+1}P_{n+2} - P_nP_{n-1} \end{aligned}$$

Lemma 2.6

$$P_{n+2}^2 - P_n^2 = 2P_{2n+2}. \quad (2.6)$$

Proof Applying (2.3)

$$\begin{aligned} P_{n+2}^2 - P_n^2 &= P_{2n+3} - P_{n+1}^2 - P_n^2 \\ &= P_{2n+3} - (P_{n+1}^2 + P_n^2) \\ &= P_{2n+3} - P_{2n+1} \\ &= 2P_{2n+2} \end{aligned}$$

A. *Reciprocal Sum of* $P_k P_{k+1}$

Lemma 2.7 If $n \geq 1$,

$$2P_{2n+1}^2 + 1 > 2(P_{n+1}^2 + 1)^2 > P_n P_{n+1} (2P_{n+1}^2 + 1) \quad (2.7)$$

Proof applying (2.3)

$$2P_{2n+1}^2 + 1 = 2(P_n^2 + P_{n+1}^2)^2 + 1 > 2P_{n+1}^4 + 4P_{n+1}^2 + 2 = 2(P_{n+1}^2 + 1)^2,$$

Therefore

$$2P_{2n+1}^2 + 1 > 2(P_{n+1}^2 + 1)^2 > 2(P_{n+1}^2 + 1)P_n P_{n+1} > P_n P_{n+1} (2P_{n+1}^2 + 1).$$

Then

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} \right)^{-1} \right] = \begin{cases} 2P_n^2, & \text{if } n \text{ is even} \\ 2P_n^2 - 1, & \text{if } n \text{ is odd} \end{cases}$$

Proof applying (2.1)

$$\frac{1}{2P_k^2} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2} = \frac{P_{k+1}^2 - 2P_k P_{k+1} - P_k^2}{2P_k^2 P_{k+1}^2} = \frac{P_{k-1}P_{k+1} - P_k^2}{2P_k^2 P_{k+1}^2} = \frac{(-1)^k}{2P_k^2 P_{k+1}^2}, \quad (2.8)$$

Therefore

$$\frac{1}{P_k P_{k+1}} = \frac{1}{2P_k^2} - \frac{1}{2P_{k+1}^2} + \frac{(-1)^{k-1}}{2P_k^2 P_{k+1}^2}, \quad (2.9)$$

Now we have

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} = \frac{1}{2P_n^2} - \frac{1}{2P_{mn+1}^2} + \sum_{k=n}^{mn} \frac{(-1)^{k-1}}{2P_k^2 P_{k+1}^2}, \quad (2.10)$$

If n is even, it is easy to see that

$$\sum_{k=n}^{mn} \frac{(-1)^{k-1}}{2P_k^2 P_{k+1}^2} < 0,$$

By the above equation (2.10)

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2}. \quad (2.11)$$

For any $k \geq 1$

$$\begin{aligned} \frac{1}{2P_k^2 + 1} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2 + 1} &= \frac{(2P_{k+1}^2 + 1)P_k P_{k+1} - (2P_k^2 + 1)(2P_{k+1}^2 + 1) - (2P_k^2 + 1)P_k P_{k+1}}{(2P_k^2 + 1)P_k P_{k+1}(2P_{k+1}^2 + 1)} \\ &= \frac{(2P_{k+1}^2 - 2P_k^2)P_k P_{k+1} - (2P_k^2 + 1)(2P_{k+1}^2 + 1)}{(2P_k^2 + 1)P_k P_{k+1}(2P_{k+1}^2 + 1)} \\ &= \frac{(-1)^k 2P_k P_{k+1} - 2P_k^2 - 2P_{k+1}^2 - 1}{(2P_k^2 + 1)P_k P_{k+1}(2P_{k+1}^2 + 1)} < 0. \end{aligned} \quad (2.12)$$

Then

$$\frac{1}{P_k P_{k+1}} = \frac{1}{2P_k^2 + 1} - \frac{1}{2P_{k+1}^2 + 1} + \frac{(-1)^{k-1} 2P_k P_{k+1} + 2P_k^2 + 2P_{k+1}^2 + 1}{(2P_k^2 + 1)P_k P_{k+1}(2P_{k+1}^2 + 1)},$$

So

$$\begin{aligned}\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} &= \frac{1}{2P_n^2 + 1} - \frac{1}{2P_{mn+1}^2 + 1} + \sum_{k=n}^{mn} \frac{(-1)^{k-1} 2P_k P_{k+1} + 2P_k^2 + 2P_{k+1}^2 + 1}{(2P_k^2 + 1)P_k P_{k+1}(2P_{k+1}^2 + 1)} \\ &> \frac{1}{2P_n^2 + 1} - \frac{1}{2P_{2n+1}^2 + 1} + \frac{(-1)^{n-1} 2P_n P_{n+1} + 2P_n^2 + 2P_{n+1}^2 + 1}{(2P_n^2 + 1)P_n P_{n+1}(2P_{n+1}^2 + 1)} \\ &> \frac{1}{2P_n^2 + 1} - \frac{1}{2P_{2n+1}^2 + 1} + \frac{1}{P_n P_{n+1}(2P_{n+1}^2 + 1)},\end{aligned}$$

Using (2.7)

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} > \frac{1}{2P_n^2 + 1}, \quad (2.13)$$

Combining (2.11) and (2.13), we have

$$\frac{1}{2P_n^2 + 1} < \sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2},$$

which means that the statement is true when n is even

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} \right)^{-1} \right] = 2P_n^2. \quad (2.14)$$

If n is odd, a similar calculation show that, for $k \geq 1$

$$\begin{aligned}\frac{1}{2P_k^2 - 1} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2 - 1} &= \frac{(2P_{k+1}^2 - 1)P_k P_{k+1} - (2P_k^2 - 1)(2P_{k+1}^2 - 1) - (2P_k^2 - 1)P_k P_{k+1}}{(2P_k^2 - 1)P_k P_{k+1}(2P_{k+1}^2 - 1)} \\ &= \frac{(2P_{k+1}^2 - 2P_k^2)P_k P_{k+1} - (2P_k^2 - 1)(2P_{k+1}^2 - 1)}{(2P_k^2 - 1)P_k P_{k+1}(2P_{k+1}^2 - 1)} \\ &= \frac{(-1)^k 2P_k P_{k+1} + 2P_k^2 + 2P_{k+1}^2 - 1}{(2P_k^2 - 1)P_k P_{k+1}(2P_{k+1}^2 - 1)} > 0,\end{aligned} \quad (2.15)$$

So

$$\frac{1}{P_k P_{k+1}} = \frac{1}{2P_k^2 - 1} - \frac{1}{2P_{k+1}^2 - 1} - \frac{(-1)^k 2P_k P_{k+1} + 2P_k^2 + 2P_{k+1}^2 - 1}{(2P_k^2 - 1)P_k P_{k+1}(2P_{k+1}^2 - 1)},$$

From which we get

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} = \frac{1}{2P_n^2 - 1} - \frac{1}{2P_{mn+1}^2 - 1} - \sum_{k=n}^{mn} \frac{(-1)^k 2P_k P_{k+1} + 2P_k^2 + 2P_{k+1}^2 - 1}{(2P_k^2 - 1)P_k P_{k+1}(2P_{k+1}^2 - 1)},$$

Hence

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2 - 1}. \quad (2.16)$$

With (2.10) and (2.6),

$$\begin{aligned}\sum_{k=n}^m \frac{1}{P_k P_{k+1}} &= \frac{1}{2P_n^2} - \frac{1}{2P_{m+1}^2} + \sum_{k=n}^m \frac{(-1)^{k-1}}{2P_k^2 P_{k+1}^2} \\ &> \frac{1}{2P_n^2} - \frac{1}{2P_{2n+1}^2} + \frac{1}{2P_n^2 P_{n+1}^2} - \frac{1}{2P_{n+1}^2 P_{n+2}^2} = \frac{1}{2P_n^2} + \frac{2P_{2n+2}}{2P_n^2 P_{n+1}^2 P_{n+2}^2} - \frac{2P_{2n+2}}{2P_{2n+1}^2 P_{n+2}^2}.\end{aligned}$$

According to (2.2) and (2.3)

$$P_{2n+1} = P_{n-1}P_{n+1} + P_n P_{n+2} = P_n^2 + P_{n+1}^2,$$

So

$$P_{2n+1} > P_n P_{n+2}, \text{ and } P_{2n+2} > P_{2n+1} > P_{n+1}^2,$$

Hence

$$2^2 P_{2n+2} P_{2n+1}^2 > 2P_n^2 P_{n+1}^2 P_{n+2}^2,$$

As a result

$$\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} > \frac{1}{2P_n^2}, \quad (2.17)$$

Combining (2.11) and (2.13),

$$\frac{1}{2P_n^2} < \sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2 - 1},$$

When n is odd, that the statement is true

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_k P_{k+1}} \right)^{-1} \right] = 2P_n^2 - 1. \quad (2.18)$$

On the basis of (2.14) and (2.18), we have the theorem 2.8.

Corollary 2.9

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}} \right)^{-1} \right] = \begin{cases} 2P_n^2, & \text{if } n \text{ is even} \\ 2P_n^2 - 1, & \text{if } n \text{ is odd} \end{cases}.$$

Proof applying (2.8), if n is even

$$\frac{1}{2P_k^2} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2} = \frac{(-1)^k}{2P_k^2 P_{k+1}^2} > 0,$$

Hence

$$\frac{1}{2P_k^2} > \frac{1}{P_k P_{k+1}} + \frac{1}{2P_{k+1}^2} > \frac{1}{P_k P_{k+1}} + \frac{1}{P_{k+1} P_{k+2}} + \frac{1}{2P_{k+2}^2} > \dots$$

So

$$\sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2}; \quad (2.19)$$

In line with (2.12)

$$\frac{1}{2P_k^2 + 1} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2 + 1} < 0$$

Therefore

$$\frac{1}{2P_k^2 + 1} < \frac{1}{P_k P_{k+1}} + \frac{1}{2P_{k+1}^2 + 1} < \frac{1}{P_k P_{k+1}} + \frac{1}{P_{k+1} P_{k+2}} + \frac{1}{2P_{k+2}^2 + 1} < \dots$$

As a result

$$\frac{1}{2P_n^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}}, \quad (2.20)$$

Combining (2.19) and (2.20), when n is even, the statement is true

$$\frac{1}{2P_n^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2}. \quad (2.21)$$

With (2.8), when n is odd,

$$\frac{1}{2P_k^2} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2} = \frac{(-1)^k}{2P_k^2 P_{k+1}^2} < 0$$

So

$$\frac{1}{2P_k^2} < \frac{1}{P_k P_{k+1}} + \frac{1}{2P_{k+1}^2} < \frac{1}{P_k P_{k+1}} + \frac{1}{P_{k+1} P_{k+2}} + \frac{1}{2P_{k+2}^2} < \dots$$

Hence

$$\frac{1}{2P_n^2} < \sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}}, \quad (2.22)$$

On the basis of (2.15)

$$\frac{1}{2P_k^2 - 1} - \frac{1}{P_k P_{k+1}} - \frac{1}{2P_{k+1}^2 - 1} > 0$$

Therefore

$$\frac{1}{2P_k^2 - 1} > \frac{1}{G_k G_{k+1}} + \frac{1}{2P_{k+1}^2 - 1} > \frac{1}{P_k P_{k+1}} + \frac{1}{P_{k+1} P_{k+2}} + \frac{1}{2P_{k+2}^2 - 1} > \dots$$

So

$$\sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2 - 1}, \quad (2.23)$$

Combining (2.22) and (2.23), when n is odd, the statement is true

$$\frac{1}{2P_n^2} < \sum_{k=n}^{\infty} \frac{1}{P_k P_{k+1}} < \frac{1}{2P_n^2 - 1}. \quad (2.24)$$

Accordance with (2.21) and (2.24), we have the Corollary 2.9.

III. MAIN RESULTS II: ALTERNATING RECIPROCAL SUMS

Lemma 3.1

$$\frac{P_{2n+2}}{P_{2n+1}} - \frac{P_{mn+2}}{P_{mn+1}} < 0. \quad (3.1)$$

Proof applying (2.1),

$$\frac{P_{2n+2}}{P_{2n+1}} - \frac{P_{mn+2}}{P_{mn+1}} = \frac{P_{2n+2} P_{mn+1} - P_{2n+1} P_{mn+2}}{P_{2n+1} P_{mn+1}} = (-1)^{2n+1} \frac{P_{(m-2)n+1}}{P_{2n+1} P_{mn+1}} < 0$$

As a similar way, we have

Lemma 3.2

$$\frac{P_{2n+1}}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}} < 0. \quad (3.2)$$

Lemma 3.3

$$\frac{1}{P_{2n} - 1} - \frac{P_{n+1}}{P_n P_{2n+1}} = \frac{(-1)^{n+1} P_n + P_{n+1}}{(P_{2n} - 1) P_n P_{2n+1}} > 0. \quad (3.3)$$

Lemma 3.4

$$\frac{1}{P_{2n}+1} - \frac{P_{n+1}}{P_n P_{2n+1}} = \frac{(-1)^{n+1} P_n - P_{n+1}}{(P_{2n}-1) P_n P_{2n+1}} < 0 \quad (3.4)$$

Lemma 3.5

$$\frac{P_{n+1}}{P_n} - \frac{1}{P_{2n}} = \frac{P_{2n+1} + (-1)^{n+1} - 1}{P_{2n}} \quad (3.5)$$

Proof combining (2.1), (2.2) and (2.3),

$$\begin{aligned} \frac{P_{n+1}}{P_n} - \frac{1}{P_{2n}} &= \frac{P_{2n} P_{n+1} - P_n}{P_n P_{2n}} = \frac{(P_{n-1} P_n + P_n P_{n+1}) P_{n+1} - P_n}{P_n P_{2n}} = \frac{P_{n-1} P_{n+1} + P_{n+1}^2 - 1}{P_{2n}} \\ &= \frac{(P_{n-1} P_{n+1} - P_n^2) + (P_n^2 + P_{n+1}^2) - 1}{P_{2n}} = \frac{P_{2n+1} + (-1)^n - 1}{P_{2n}}. \end{aligned}$$

Use the same way, we have

Lemma 3.6

$$\frac{P_{n+1}}{P_n} + \frac{1}{P_{2n}} = \frac{P_{2n+1} + (-1)^n + 1}{P_{2n}} \quad (3.6)$$

Theorem

$$\left[\left(\sum_{k=n}^{mn} \frac{(-1)^k}{P_k P_{k+1}} \right)^{-1} \right] = \begin{cases} P_{2n} - 1, & \text{if } n \text{ is even} \\ -P_{2n} - 1, & \text{if } n \text{ is odd} \end{cases} \quad 3.7$$

Proof applying (2.1),

$$\frac{(-1)^k}{P_k P_{k+1}} = \frac{P_{k+1}^2 - P_k P_{k+2}}{P_k P_{k+1}} = \frac{P_{k+1}}{P_k} - \frac{P_{k+2}}{P_{k+1}},$$

So

$$\sum_{k=n}^{mn} \frac{(-1)^k}{P_k P_{k+1}} = \sum_{k=n}^{mn} \left(\frac{P_{k+1}}{P_k} - \frac{P_{k+2}}{P_{k+1}} \right) = \frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}}, \quad (3.7)$$

If n is even, with (3.1) and (2.1)

$$\frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} < \frac{P_{n+1}}{P_n} - \frac{P_{2n+2}}{P_{2n+1}} = \frac{P_{2n+1} P_{n+1} - P_{2n+2} P_n}{P_n P_{2n+1}} = \frac{(-1)^n P_{n+1}}{P_n P_{2n+1}} = \frac{P_{n+1}}{P_n P_{2n+1}},$$

On the basis of (3.3)

$$\frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} < \frac{P_{n+1}}{P_n P_{2n+1}} < \frac{1}{P_{2n} - 1}; \quad (3.8)$$

According to (3.5) and (3.2)

$$\begin{aligned} \frac{P_{n+1}}{P_n} - \frac{1}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}} &= \frac{P_{2n+1}}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}} + \frac{(-1)^{n+1} - 1}{P_{2n}} = \frac{P_{2n+1}}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}}; \\ \frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} &> \frac{1}{P_{2n}}, \end{aligned} \quad (3.9)$$

Combining (3.7), (3.8) and (3.9), when n is even, the statement is true

$$\left[\left(\sum_{k=n}^{mn} \frac{(-1)^k}{P_k P_{k+1}} \right)^{-1} \right] = P_{2n} - 1 \quad (3.10)$$

If n is even, with (3.1) and (2.1)

$$\frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} < \frac{P_{n+1}}{P_n} - \frac{P_{2n+2}}{P_{2n+1}} = \frac{P_{2n+1} P_{n+1} - P_{2n+2} P_n}{P_n P_{2n+1}} = \frac{(-1)^n P_{n+1}}{P_n P_{2n+1}} = -\frac{P_{n+1}}{P_n P_{2n+1}},$$

Applying (3.4)

$$\frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} < -\frac{1}{P_{2n} + 1}, \quad (3.11)$$

Using (3.6) and (3.2)

$$\frac{P_{n+1}}{P_n} + \frac{1}{P_{2n}} = \frac{P_{2n+1} + (-1)^{n+1} + 1}{P_{2n}} = \frac{P_{2n+1}}{P_{2n}},$$

$$\frac{P_{n+1}}{P_n} + \frac{1}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}} = \frac{P_{2n+1}}{P_{2n}} - \frac{P_{mn+2}}{P_{mn+1}} > 0,$$

Hence

$$\frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} > -\frac{1}{P_{2n}}, \quad (3.12)$$

Combining (3.7), (3.11) and (3.12), when n is odd, the statement is true

$$-\frac{1}{P_{2n}} < \sum_{k=n}^{mn} \frac{(-1)^k}{P_k P_{k+1}} = \frac{P_{n+1}}{P_n} - \frac{P_{mn+2}}{P_{mn+1}} < -\frac{1}{P_{2n} + 1},$$

Therefore

$$\left[\left(\sum_{k=n}^{mn} \frac{(-1)^k}{P_k P_{k+1}} \right)^{-1} \right] = -P_{2n} - 1 \quad (3.13)$$

According to (3.10) and (3.13), we established the theorem 3.7.

IV. MAIN RESULTS III: THE RECIPROCAL SUMS OF EVEN AND ODD TERMS

Theorem 4.1 If $m \geq 3$ and $n \geq 1$,

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_{2k}} \right)^{-1} \right] = 2P_{2n-1} - 1$$

Proof applying (2.1), or $k \geq 2$

$$\frac{1}{2P_{2k-1}-1} - \frac{1}{P_{2k}} - \frac{1}{2P_{2k+1}-1} = \frac{2(P_{2k+1}-P_{2k-1})P_{2k} - (2P_{2k-1}-1)(2P_{2k+1}-1)}{(2P_{2k-1}-1)P_{2k}(2P_{2k+1}-1)} = \frac{2P_{2k-1}+2P_{2k+1}-5}{(2P_{2k-1}-1)P_{2k}(2P_{2k+1}-1)} > 0$$

$$\frac{1}{2P_{2k-1}-1} - \frac{1}{P_{2k}} - \frac{1}{2P_{2k+1}-1} = \frac{2P_{2k-1}+2P_{2k+1}-5}{(2P_{2k-1}-1)P_{2k}(2P_{2k+1}-1)} > 0 \quad (4.1)$$

Which implies that

$$\frac{1}{P_{2k}} = \frac{1}{2P_{2k-1}-1} - \frac{1}{2P_{2k+1}-1} - \frac{2P_{2k-1}+2P_{2k+1}-5}{(2P_{2k-1}-1)P_{2k}(2P_{2k+1}-1)} < \frac{1}{2P_{2k-1}-1} - \frac{1}{2P_{2k+1}-1},$$

So

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}} < \frac{1}{2P_{2n-1}-1} - \frac{1}{2P_{2mn+1}-1} < \frac{1}{2P_{2n-1}-1} \quad (4.2)$$

From (2.1) we obtain

$$\frac{1}{2P_{2k-1}} - \frac{1}{P_{2k}} - \frac{1}{2P_{2k+1}} = \frac{P_{2k+1}P_{2k} - 2P_{2k-1}P_{2k+1} - P_{2k-1}P_{2k}}{2P_{2k-1}P_{2k}P_{2k+1}} = \frac{-1}{P_{2k-1}P_{2k}P_{2k+1}} \quad (4.3)$$

Hence

$$\frac{1}{P_{2k}} = \frac{1}{2P_{2k-1}} - \frac{1}{2P_{2k+1}} + \frac{1}{P_{2k-1}P_{2k}P_{2k+1}},$$

Therefore

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}} = \frac{1}{2P_{2n-1}} - \frac{1}{2P_{2mn+1}} + \sum_{k=n}^{mn} \frac{1}{P_{2k-1}P_{2k}P_{2k+1}} > \frac{1}{2P_{2n-1}} - \frac{1}{2P_{2mn+1}} + \frac{1}{P_{2n-1}P_{2n}P_{2n+1}} \quad (4.4)$$

For all $n \geq 1, m \geq 3$, with (2.2)

$$P_{m-1}P_n + P_mP_{n+1} = P_{m+n}, \quad P_{m+n} > P_mP_{n+1} > P_mP_n,$$

Which implies that

$$P_{2n-1}P_{2n}P_{2n+1} < P_{6n+1} < P_{2mn+1},$$

With (4.4)

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}} > \frac{1}{2P_{2n-1}} - \frac{1}{2P_{2mn+1}} + \frac{1}{P_{2n-1}P_{2n}P_{2n+1}} > \frac{1}{2P_{2n-1}} \quad (4.5)$$

According to (4.2) and (4.5), we established the theorem 4.1.

Corollary.4.2

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{P_{2k}} \right)^{-1} \right] = 2P_{2n-1} - 1.$$

Theorem 4.3 If $m \geq 2$ and $n \geq 1$,

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} \right)^{-1} \right] = 2P_{2n-2}.$$

Proof applying (2.1), or $k \geq 2$

$$\frac{1}{2P_{2k-2}} - \frac{1}{P_{2k-1}} - \frac{1}{2P_{2k}} = \frac{P_{2k-1}P_{2k} - 2P_{2k-2}P_{2k} - P_{2k-2}P_{2k-1}}{2P_{2k-2}P_{2k-1}P_{2k}} = \frac{1}{P_{2k-1}P_{2k}P_{2k+1}} > 0, \quad (4.6)$$

$$\frac{1}{P_{2k-1}} = \frac{1}{2P_{2k-2}} - \frac{1}{2P_{2k}} - \frac{1}{P_{2k-2}P_{2k-1}P_{2k}} < \frac{1}{2P_{2k-2}} - \frac{1}{2P_{2k}},$$

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} < \frac{1}{2P_{2n-2}} - \frac{1}{2P_{2mn}} < \frac{1}{2P_{2n-2}} \quad (4.7)$$

$$\frac{1}{2P_{2k-2}+1} - \frac{1}{P_{2k-1}} - \frac{1}{2P_{2k}+1} = \frac{2(P_{2k}-P_{2k-2})P_{2k-1} - (2P_{2k-2}+1)(2P_{2k}+1)}{(2P_{2k-2}+1)P_{2k-1}(2P_{2k}+1)} = \frac{2^2-2P_{2k-2}-2P_{2k}-1}{(2P_{2k-2}+1)P_{2k-1}(2P_{2k}+1)} < 0$$

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} = \frac{1}{2P_{2n-2}+1} - \frac{1}{2P_{2mn}+1} + \sum_{k=n}^{mn} \frac{2P_{2k-2}+2P_{2k}+1-2^2}{(2P_{2k-2}+1)P_{2k-1}(2P_{2k}+1)},$$

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} > \frac{1}{2P_{2n-2}+1} - \frac{1}{2P_{2mn}+1} + \frac{2P_{2n-2}+2P_{2n}+1-2^2}{(2P_{2n-2}+1)P_{2n-1}(2P_{2n}+1)},$$

By reason of

$$\frac{2P_{2n-2}+2P_{2n}+1-2^2}{(2P_{2n-2}+1)P_{2n-1}(2P_{2n}+1)} > \frac{2P_{2n}+1}{(2P_{2n-2}+1)P_{2n-1}(2P_{2n}+1)} > \frac{1}{(2P_{2n-2}+1)P_{2n-1}} > \frac{1}{2P_{4n}+1} > \frac{1}{2P_{2mn}+1},$$

Hence

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} > \frac{1}{2P_{2n-2}+1} \quad (4.8)$$

According to (4.7) and (4.8), the theorem 4.3 is true.

$$\text{Corollary 4.4} \quad \left[\left(\sum_{k=n}^{\infty} \frac{1}{P_{2k-1}} \right)^{-1} \right] = 2P_{2n-2}.$$

Theorem 4.5 If $m \geq 2$ and $n \geq 1$,

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_{2k}^2} \right)^{-1} \right] = 2P_{4n-2} - 1.$$

Proof applying (2.1) and (2.4)

$$P_{2k}^2 - P_{2k-2}^2 = 2P_{4k-2}, \quad (4.9)$$

$$P_{2k+2}^2 - P_{2k}^2 = 2P_{4k+2}, \quad (4.10)$$

$$P_{2k-1}^2 P_{2k+1}^2 - P_{2k-2}^2 P_{2k+2}^2 = 5(P_{2k}^2 + P_{2k-1} P_{2k+1} - 4), \quad (4.11)$$

According to (4.9), (4.10) and (4.11)

$$\frac{1}{2P_{2k-2}} - \frac{1}{P_{2k-1}} - \frac{1}{2P_{2k}} = \frac{P_{2k-1}P_{2k} - 2P_{2k-2}P_{2k} - P_{2k-2}P_{2k-1}}{2P_{2k-2}P_{2k-1}P_{2k}} = \frac{1}{P_{2k-1}P_{2k}P_{2k+1}} > 0$$

Hence

$$\frac{1}{2P_{4k-2}-1} - \frac{1}{P_{2k}^2} - \frac{1}{2P_{4k+2}-1} > 0, \quad (4.12)$$

Therefore

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}^2} < \frac{1}{2P_{4n-2}-1} - \frac{1}{2P_{4mn+2}-1} < \frac{1}{2P_{4n-2}-1} \quad (4.13)$$

For all $k \geq 2$,

$$\begin{aligned} \frac{1}{2P_{4k-2}} - \frac{1}{P_{2k}^2} - \frac{1}{2P_{4k+2}} &= \frac{P_{2k}^2(P_{4k+2} - P_{4k-2}) - 2P_{4k-2}P_{4k+2}}{2G_{4k-2}G_{2k}^2G_{4k+2}} \\ &= \frac{-(4P_{2k}^2 - 4P_{2k-1}P_{2k+1})}{P_{4k-2}P_{2k}^2P_{4k+2}} < \frac{-1}{P_{4k-2}P_{4k+2}} < 0 \end{aligned} \quad (4.14)$$

From which we obtain

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}^2} > \frac{1}{2P_{4n-2}} - \frac{1}{2P_{4mn+2}} + \sum_{k=n}^{mn} \frac{1}{P_{4k-2}P_{4k+2}} > \frac{1}{2P_{4n-2}} - \frac{1}{2P_{4mn+2}} + \frac{1}{P_{4n-2}P_{4n+2}}$$

Applying (2.4)

$$2P_{8n+2} = P_{4n+2}^2 - P_{4n}^2 = 2P_{4n+1}(P_{4n+2} + P_{4n}) > 2P_{4n+1}P_{4n+2} > 2P_{4n+2}P_{4n-2}$$

So

$$\sum_{k=n}^{mn} \frac{1}{P_{2k}^2} > \frac{1}{2P_{4n-2}} \quad (4.15)$$

Combining (4.13) and (4.15), we have theorem 4.5

$$\text{Corollary 4.6} \quad \left[\left(\sum_{k=n}^{\infty} \frac{1}{P_{2k}^2} \right)^{-1} \right] = 2P_{4n-2} - 1.$$

Theorem 4.7 If $m \geq 2$ and $n \geq 1$,

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}^2} \right)^{-1} \right] = 2P_{4n-4}.$$

Proof by the calculation of (2.4), we obtain, $k \geq 2$

$$P_{2k+1}^2 - P_{2k-1}^2 = 2P_{4k} \quad (4.16)$$

$$P_{2k-1}^2 - P_{2k-3}^2 = 2P_{4k-4} \quad (4.17)$$

From (4.16) and (4.17)

$$\begin{aligned} &P_{2k-1}^2(P_{4k} - P_{4k-4}) - 2P_{4k}P_{4k-4} \\ &= \frac{1}{2}P_{2k-1}^2(P_{2k+1}^2 - 2P_{2k-1}^2 + P_{2k-3}^2) - \frac{1}{2}(P_{2k+1}^2 - P_{2k-1}^2)(P_{2k-1}^2 - P_{2k-3}^2) \\ &= \frac{1}{2}(P_{2k+1}P_{2k-3} - P_{2k-1}^2)(P_{2k+1}P_{2k-3} + P_{2k-1}^2) \\ &= 2(P_{2k+1}P_{2k-3} + P_{2k-1}^2) \end{aligned} \quad (4.18)$$

By (4.16), (4.17) and (4.18)

$$\frac{1}{2P_{4k-4}} - \frac{1}{P_{2k-1}^2} - \frac{1}{2P_{4k}} = \frac{P_{2k-1}^2(P_{4k} - P_{4k-4}) - 2P_{4k}P_{4k-4}}{2P_{4k-4}P_{2k-1}^2P_{4k}} = \frac{P_{2k+1}P_{2k-3} + P_{2k-1}^2}{P_{4k-4}P_{2k-1}^2P_{4k}} > 0,$$

Which implies

$$\frac{1}{P_{2k-1}^2} < \frac{1}{2P_{4k-4}} - \frac{1}{2P_{4k}}$$

Hence

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}^2} < \frac{1}{2P_{4n-4}} - \frac{1}{2P_{4mn}} < \frac{1}{2P_{4n-4}}, \quad (4.19)$$

$$\begin{aligned} \frac{1}{2P_{4k-4}+1} - \frac{1}{P_{2k-1}^2} - \frac{1}{2P_{4k}+1} &= \frac{2(P_{4k}-P_{4k-4})P_{2k-1}^2 - (2P_{4k-4}+1)2(aP_{4k}+1)}{(2P_{4k-4}+1)P_{2k-1}^2 2(aP_{4k}+1)} \\ &= \frac{2^2(P_{2k+1}P_{2k-3}+P_{2k-1}^2) - (P_{2k+1}^2 - P_{2k-3}^2) - 1}{(2P_{4k-4}+1)P_{2k-1}^2(2P_{4k}+1)} \end{aligned}$$

By reason of

$$P_{2k+1}P_{2k-3} + P_{2k-1}^2 = (2P_{2k} - P_{2k-1})(P_{2k-1} - 2P_{2k-2}) + P_{2k-1}^2 = 5P_{2k-1}^2 + 4,$$

$$P_{2k+1}^2 - P_{2k-3}^2 = 12P_{2k-1}(P_{2k} + P_{2k-2}) > 24P_{2k-1}^2 + 16,$$

Therefore

$$2^2(P_{2k+1}P_{2k-3} + P_{2k-1}^2) - (P_{2k+1}^2 - P_{2k-3}^2) - 1 < -P_{2k-1}^2.$$

So

$$\frac{1}{2P_{4k-4}+1} - \frac{1}{P_{2k-1}^2} - \frac{1}{2P_{4k}+1} < \frac{-1}{(2P_{4k-4}+1)(2P_{4k}+1)} < 0 \quad (4.20)$$

$$\frac{1}{P_{2k-1}^2} > \frac{1}{2P_{4k-4}+1} - \frac{1}{2P_{4k}+1} + \frac{1}{(2P_{4k-4}+1)(2P_{4k}+1)},$$

$$\begin{aligned} \sum_{k=n}^{mn} \frac{1}{P_{2k-1}^2} &> \frac{1}{2P_{4n-4}+1} - \frac{1}{2P_{4mn}+1} + \sum_{k=n}^{mn} \frac{1}{(2P_{4k-4}+1)(2P_{4k}+1)} \\ &> \frac{1}{2P_{4n-4}+1} - \frac{1}{2P_{4mn}+1} + \frac{1}{(2P_{4n-4}+1)(2P_{4n}+1)}, \end{aligned}$$

Also because

$$(2P_{4n-4}+1)(2P_{4n}+1) = 2^2P_{4n-4}P_{4n} + 2P_{4n-4} + 2P_{4n} + 1 < P_{8n} + 1.$$

So

$$\sum_{k=n}^{mn} \frac{1}{P_{2k-1}^2} > \frac{1}{2P_{4n-4}+1} \quad (4.21)$$

Combining (4.19) and (4.21), the statement is true

$$\left[\left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}^2} \right)^{-1} \right] = 2P_{4n-4}$$

$$\text{Corollary 4.8} \quad \left[\left(\sum_{k=n}^{\infty} \frac{1}{P_{2k-1}^2} \right)^{-1} \right] = 2P_{4n-4}.$$

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