

Fiducial Region Estimation of Parameter of Two Parameters Exponential Distribution

Xiuzhen Li¹, Yanying Ma^{1,a*}, Chunguang Huang² and Xin Wang³

¹Jilin Engineering Normal University .Changchun; China;

²NO.11 High school of Changchun Jilin China

³Shenyang Jianzhu University. Shenyang, China

^a2238552865@qq.com

* corresponding author

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Abstract. Exponential distribution is very important distribution, it is often used to approximate various life distributions, and is widely used in the areas of queuing theory and reliability theory. The article studied the combination fiducial region estimation of two parameters exponential distribution under the study of fiducial interval estimation of single parameter of two parameters exponential distribution.

Introduction

Definition of two parameters exponential distribution

Definition: Suppose the random variable x obey the two parameters exponential distribution. Its distribution function and distribution density function is

$$F(x; \mu, \sigma) = 1 - e^{-\frac{x-\mu}{\sigma}}; \quad f(x; \mu, \sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}$$

Which $x \geq \mu, \sigma > 0, \mu$ is positional parameter, σ is scale parameter, then we called the random variable x obey the two parameters exponential distribution. often note: $x \sim E(\mu, \sigma)$

Fundamental lemma

Suppose X_1, X_2, \dots, X_n are Simple random samples with capacity of n that from exponential distribution, then we have order statistic $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$, obtained sample observations x_1, x_2, \dots, x_n , and then the maximum likelihood estimation of μ and σ^2 are obtained

$$\hat{\mu} = X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$$

$$\hat{\sigma} = \bar{X} - X_{(1)}$$

In order to study the problem, convenience, the following four lemmas are given

Lemma 1: Suppose $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ is the first r order statistics ($1 \leq r \leq n$) that from population $E(\mu, \sigma)$, then

$$Y = \frac{2n(X_{(1)} - \mu)}{\sigma} \sim \chi^2 \quad (1)$$

Lemma 2: Suppose $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ is the first r order statistics that from population $E(\mu, \sigma)$ that from population $E(\mu, \sigma)$

Remember $Y_1 = nX_{(1)}, Y_2 = (n-1)(X_{(2)} - X_{(1)}), \dots, Y_i = (n-i+1)(X_{(i)} - X_{(i-1)}), i=2, 3, \dots, n$, then We have the following conclusions

$$\textcircled{1} \sum_{i=1}^n X_{(i)} = \sum_{i=1}^n Y_i$$

② Y_1, Y_2, \dots, Y_n independent of each other (2)

③ $Y_1 \sim E(n\mu_n\sigma), Y_2, Y_3 \dots Y_n$ are the same as single parameter exponential distribution $E(\sigma)$

Lemma 3: Suppose Z_1, Z_2, \dots, Z_n independent and the same distribution as $E(\sigma)$, then

$$Z_1 + Z_2 + \dots + Z_n \sim Ga(n, \frac{1}{\sigma}) \quad (3)$$

Lemma 4: Suppose $Z \sim Ga(n, \frac{1}{\sigma})$, then $\frac{2Z}{\sigma} \sim \chi^2(2n)$. (4)

Building function model, Induced fiducial distribution

Extract a simple random sample X_1, X_2, \dots, X_n from the population $E(\mu, \sigma)$, obtain the first r order statistics $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ ($1 \leq r \leq n$). By Lemma 1 we can know

$$\frac{2n(X_{(1)} - \mu)}{\sigma} \sim \chi^2(2)$$

Note $Y_i = (n-i+1)(X_{(i)} - X_{(i-1)})$ $i=2, 3, \dots, n$. then

$$n(\bar{X} - X_{(1)}) = \sum_{i=1}^n X_{(i)} - nX_{(1)} = \sum_{i=1}^n Y_i - Y_1 = \sum_{i=2}^n Y_i$$

By Lemma 2 we can know: Y_2, \dots, Y_n is equally distributed in Single parameter exponential distribution $E(\sigma)$, then by Lemma 3 we can obtain:

$$n(\bar{X} - X_{(1)}) = \sum_{i=2}^n Y_i = Y_2 + Y_3 + \dots + Y_n \sim Ga(n-1, \frac{1}{\sigma}) \quad (5)$$

$$\text{Also due to } \frac{2}{\sigma} \sum_{i=2}^n Y_i = \frac{2n(\bar{X} - X_{(1)})}{\sigma} \quad (6)$$

$$\text{By Lemma 4 we can obtain: } \frac{2n(\bar{X} - X_{(1)})}{\sigma} \sim \chi^2(2(n-1)) \quad (7)$$

$$\text{Note: } \begin{cases} e_1 = \frac{2n(X_{(1)} - \mu)}{\sigma} \sim \chi^2(2) \\ e_2 = \frac{2n(\bar{X} - X_{(1)})}{\sigma} \sim \chi^2(2(n-1)) \end{cases} \quad (8)$$

Because e_1 is a function of $X_{(1)}$, e_2 is a function of $(\bar{X} - X_{(1)})$. yet $X_{(1)}$ and $(\bar{X} - X_{(1)})$ independent of each other, so e_1 and e_2 independent of each other too, so we can obtain function model:

$$\begin{cases} X_{(1)} = \frac{\sigma}{2n} e_1 + \mu \\ \bar{X} - X_{(1)} = \frac{\sigma}{2n} e_2 \end{cases} \quad (9)$$

In this function model, $(X_{(1)}, \bar{X} - X_{(1)})$ is sample observations, (μ, σ) is parameter vector, (e_1, e_2) is error variance, so by (9) we can obtained the following conclusion

$$\begin{cases} \mu = X_{(1)} - \frac{e_1}{e_2} (\bar{X} - X_{(1)}) \\ \sigma = \frac{2n(\bar{X} - X_{(1)})}{e_2} \end{cases} \quad (10)$$

Because e_1, e_2 is independent of each other, by (8) we can obtain joint density function of (e_1, e_2) is

$$\begin{aligned}
 h(e_1, e_2) &= \frac{1}{2} \exp \left\{ -\frac{1}{2} e_1 \right\} \frac{1}{2^{n-1} \Gamma(n-1)} e_2^{n-2} \exp \left\{ -\frac{1}{2} e_2 \right\} \\
 &= \frac{1}{2^n \Gamma(n-1)} e_2^{n-2} \exp \left\{ -\frac{1}{2} (e_1 + e_2) \right\}
 \end{aligned} \quad (11)$$

According to the transformation formula of random vector, Do transform from (e_1, e_2) to (μ, σ) , Find the absolute value of the Jacobi determinant corresponding to (8)

$$|J| = \begin{vmatrix} \frac{\partial e_1}{\partial \mu} & \frac{\partial e_1}{\partial \sigma} \\ \frac{\partial e_2}{\partial \mu} & \frac{\partial e_2}{\partial \sigma} \end{vmatrix} = \begin{vmatrix} -2n & -2n(x_{(1)} - \mu) \\ \sigma & \frac{\sigma^2}{\sigma^2} \\ 0 & -2n(\bar{x} - x_{(1)}) \\ & \sigma^2 \end{vmatrix} = \left| \frac{4n^2(\bar{x} - x_{(1)})}{\sigma^3} \right|$$

By bringing in operations we can obtain the joint fiducial density function of (μ, σ)

$$\begin{aligned}
 g(\mu, \sigma) &= \exp \left\{ -\frac{n(\bar{x} - \mu)}{\sigma} \right\} \frac{1}{2^{n-2} \Gamma(n-1)} \left[\frac{2n(\bar{x} - x_{(1)})}{\sigma} \right]^{n-2} \cdot \frac{n^2(\bar{x} - x_{(1)})}{\sigma^3} \\
 &= \exp \left\{ -\frac{n(\bar{x} - \mu)}{\sigma} \right\} \frac{1}{\Gamma(n-1)} \cdot \frac{n^n(\bar{x} - x_{(1)})^{n-1}}{\sigma^{n+1}} \quad \mu < x_{(1)}, \quad \sigma > 0
 \end{aligned} \quad (12)$$

On the open interval $(-\infty, x_{(1)})$ The last formula for μ integral, the marginal fiducial density function of σ can be obtained:

$$\begin{aligned}
 f(\sigma) &= \int_{-\infty}^{x_{(1)}} g(\mu, \sigma) d\mu \\
 &= \exp \left\{ -\frac{n(\bar{x} - x_{(1)})}{\sigma} \right\} \frac{\left[n(\bar{x} - x_{(1)}) \right]^{n-1}}{\Gamma(n-1)} \cdot \sigma^{-n} \quad \sigma > 0
 \end{aligned} \quad (13)$$

The same methods, the marginal fiducial density function of μ can be obtained

$$\begin{aligned}
 \tau(\mu) &= \int_0^\infty g(\mu, \sigma) d\sigma \\
 &= (n-1) \frac{(\bar{x} - x_{(1)})^{n-1}}{(\bar{x} - \mu)^n} \quad (\mu < x_{(1)})
 \end{aligned} \quad (14)$$

The estimation of fiducial region (interval) is given

According to (12), For a given level of $1-\alpha$ ($0 < \alpha < 1$), Existence of G makes $\iint_G g(\mu, \sigma) d\mu d\sigma = 1-\alpha$

Then the G is a joint fiducial region estimation of the two parameters (μ, σ) , that the fiducial level is $1-\alpha$ ($0 < \alpha < 1$), Obviously, there's more than one of these G , The following is a very special case. Although μ and σ are not necessarily independent, but $e_1 \sim \chi^2(2)$, $e_2 \sim \chi^2(2(n-1))$ Moreover e_1, e_2 is independent of each other

$$e_1 = \frac{2n(X_{(1)} - \mu)}{\sigma}, \quad e_2 = \frac{2n(\bar{X} - X_{(1)})}{\sigma}$$

For a given fiducial level $1-\alpha$, take the appropriate a_1, a_2, b_1, b_2 then

$P(a_1 < e_1 < a_2, b_1 < e_2 < b_2) = 1-\alpha$ because e_1, e_2 independent of each other, then

$P(a_1 < e_1 < a_2)P(b_1 < e_2 < b_2) = 1-\alpha$

Arbitrary selection $0 < \beta_1 < 1, 0 < \beta_2 < 1$, make $\beta_1 \beta_2 = 1-\alpha$

Make $P(a_1 < e_1 < a_2) = \beta_1, P(b_1 < e_2 < b_2) = \beta_2$

Because e_1 and e_2 obeys χ^2 distribution, The distribution density images are unimodal and asymmetric, in order to study the problem, convenience, Take the tail case (Note: It's also another the reason why the region of fiducial that is chosen here is not unique). Take G's rectangle about (e_1, e_2)

$$\text{Take } a_1 = \chi_{\frac{1-\beta_1}{2}}^2(2), a_2 = \chi_{\frac{1+\beta_1}{2}}^2(2); \quad b_1 = \chi_{\frac{1-\beta_2}{2}}^2(2(n-1)), b_2 = \chi_{\frac{1+\beta_2}{2}}^2(2(n-1))$$

because

$$P(a_1 < e_1 < a_2, b_1 < e_2 < b_2)$$

$$= P\left\{ \frac{a_2\sigma}{2n} - X_{(1)} < \mu < \frac{a_1\sigma}{2n} - X_{(1)}, \quad \frac{2n(\bar{X} - X_{(1)})}{b_2} < \sigma < \frac{2n(\bar{X} - X_{(1)})}{b_1} \right\}$$

so, a fiducial region G of (μ, σ) that fiducial level is $1 - \alpha$ ($0 < \alpha < 1$)

$$G = \left\{ \frac{a_2\sigma}{2n} - X_{(1)} < \mu < \frac{a_1\sigma}{2n} - X_{(1)}, \quad \frac{2n(\bar{X} - X_{(1)})}{b_2} < \sigma < \frac{2n(\bar{X} - X_{(1)})}{b_1} \right\}$$

Where a_1, a_2, b_1, b_2 is optional constant, $\beta_1\beta_2 = 1 - \alpha$.

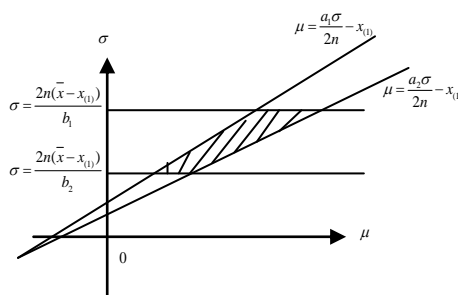


Figure 1. Fiducial region

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References

- [1] Mao Shi-song, Wang Jing-long, Pu Xiao-long. Advanced Mathematical Statistics [M]. higher education publishing house, 1998
- [2] zhang Hong-bing, Liu Rui-yuan; Li jie. The shortest confidence interval of parameters of two parameter exponential distribution [J]. Journal of Neijiang Normal University, 2007, 12.
- [3] Gao Shang. Calculation of shortest confidence interval [J]. Journal of east china shipbuilding institute, 2003.
- [4] Dawid A P and Stone M. The functional-model basis of fiducial inference (with discussion). Ann Statist, 1982, 10: 1054-1074
- [5] Dawid AP and Wang J L. Fiducial prediction and semi-bayesian inference. Ann Statist, 1993, 21: 1119-1138.
- [6] Chen Xi-ru. Probability Theory and Mathematical Statistics [M]. University of Science and Technology of China publishing house, 1992.