The Product and Covering of Lattice–Valued Tree Automata
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Abstract. In this paper, the concept of full direct product, restricted direct product, cascade product, wreath product and covering is given. The relation between the products of the lattice-valued tree automata, the covering relation between the lattice-value tree automata, and the covering relation between the product of lattice-valued tree automata and the product of the lattice-valued tree automata covering them are discussed.

Introduction

The tree automata can be regarded as the generalization of the classical word automata[1]. The systematic exposition of the classical tree automata can be found in literature[2]. Li Y M and Pedrycz W put forward the theory of lattice-valued automata, which constructed the fuzzy automata in a broader framework. Literature[3] is the generalization of the key concepts and conclusions of the literature[4,5]. In this paper, literature[6] studies the congruence and homomorphism of lattice-valued tree automata. In automata theory, product is one of the basic operations, and the product and covering relation of different forms play a very important role in the decomposition of automata. Literature [7] studies the product of lattice-valued finite automata, and literature [8] studies the product and covering relation of fuzzy finite-state machines. In this paper, the concept of product and covering of the lattice-valued tree automata is given, the covering relation between the products of the lattice-valued tree automata is studied, and the covering relation between the product of lattice-valued automata and the product of the lattice-valued tree automata covering them is also discussed.

Basic Concepts and Symbols

Definition 2.1[9]. Assuming that $\Sigma$ is a nonempty set, $rk: \Sigma \rightarrow N$ is a mapping, and $N$ is the set of natural numbers, then $(\Sigma, rk)$ is called an order character set. $\forall k \geq 0$, $\Sigma(k) = \{\sigma \in \Sigma | rk(\sigma) = k\}$. For simplicity, $(\Sigma, rk)$ is recorded as $\Sigma$.

Definition 2.2[10]. Assuming that $X$ is a set of variables that does not cross with $\Sigma$. The minimum set that meets the condition (1) and (2) is called the item set on the $\Sigma$ marked by $X$, and it is recorded as $T_\Sigma(X)$. Among them:

$X \cup \Sigma(0) \subseteq T_\Sigma(X)$

If $k \geq 1, \sigma \in \Sigma(k), s_1, \ldots, s_k \in T_\Sigma(X)$, then $\sigma(s_1, \ldots, s_k) \in T_\Sigma(X)$.

Note: The set $T_\Sigma(\emptyset)$ is recorded as $T_\Sigma$. Obviously, $T_\Sigma(X) = T_{\Sigma \cup X}$.

Definition 2.3[4]. We give a lattice $(L, \land, \lor, 0, 1)$, and let $\bullet$ be a binary operation in $L$, and make $(L, \bullet, e)$ be a monoid with a unit element $e$. Any $a, b, c, x \in L$, and meets the following conditions:

$a \bullet 0 = 0 \bullet a = 0$

$a \leq b \Rightarrow a \bullet x \leq b \bullet x \land x \bullet a \leq x \bullet b$

$a \bullet (b \lor c) = (a \bullet b) \lor (a \bullet c),(b \lor c) \bullet a = (b \bullet a) \lor (c \bullet a)$

Then $L$ is called a lattice-ordered semigroup, and is called a lattice semigroup for short. For
simplicity, it is generally recorded as \( L \).

**Definition 2.4** [11]. A lattice-valued finite tree automaton \( M = (Q, \Sigma, L, \alpha, \nu) \). Among them, \( Q \) is the finite nonempty state set, \( \Sigma \) is the order input character set, \( L \) is a lattice semigroup, \( \nu : Q \to L \) is the fuzzy termination state, \( \alpha \) is a lattice-valued tree representation, which is a family mapping \( \alpha = (\alpha_k)_{k \geq 0}, \alpha_k : \Sigma_k \to L^Q \). For simplicity, the multiplicative operation \( \cdot \) of lattice semigroup \( L \) is expressed as \( \otimes \).

Let’s define the function \( \alpha_k : Q^k \times Q \to L, k \in \mathbb{N}, \forall t = \sigma(t_1, \cdots, t_k) \in T_k \),

\[
\alpha_t(p, p_2, \cdots, p_k) = \alpha_{\sigma(t_1, \cdots, t_k)}(p, p_2, \cdots, p_k) \equiv \bigvee_{t_i \in Q, i = 1, \cdots, k} \alpha_t(p, p_2, \cdots, p_k, p) \otimes \alpha_t_p(\sigma(t_1, \cdots, t_k))_p.
\]

**Definition 2.5** [6]. Assuming that \( M_1 = (Q_1, \Sigma_1, L_1, \alpha_1, \nu_1) \) and \( M_2 = (Q_2, \Sigma_2, L_2, \alpha_2, \nu_2) \) are two lattice-valued tree automata. Mapping \( \varphi : Q_1 \to Q_2 \) is called the homomorphic mapping from \( M_1 \) to \( M_2 \), which is recorded as \( \varphi : M_1 \to M_2 \). If it meets:

\[
\alpha_{k_1}(\sigma)(p_1, \cdots, p_k) \leq \alpha_{k_2}(\sigma)(p_1, \cdots, p_k)_p, \quad \nu_1(p) \leq \nu_2(\varphi(p)), \quad \forall p, p_i \in Q_i, i \in [1, k], \sigma \in \Sigma_k, k \geq 0
\]

\( \varphi \) is called a strong homomorphism, if:

\[
\alpha_{k_2}(\sigma)(p_1, \cdots, p_k)_p = \nu_2(\varphi(p)), \quad \nu_1(p) \leq \nu_2(\varphi(p)), \quad \forall p, p_i \in Q_i, i \in [1, k], \sigma \in \Sigma_k, k \geq 0
\]

A strong homomorphism \( \varphi : M_1 \to M_2 \) is called an isomorphism. If \( \varphi \) is the bijection; If \( \varphi \) is the injection/ surjection, then we say \( \varphi \) is the monomorphism mapping (epimorphism mapping).

**The Covering of the Lattice-Valued Tree Automata**

**Definition 3.1**. Assuming that \( M_i = (Q_i, \Sigma_i, L, \alpha_i, \nu_i)(i = 1, 2) \) is the lattice-valued tree automaton. If there is the surjection \( \varphi : Q_2 \to Q_1 \), and preserving rank \( \psi : \Sigma_1 \to \Sigma_2 \), for \( \forall q_1, q_2, \cdots, q_k, q \in Q_2, \sigma_1 \in \Sigma_k \), it satisfies:

\[
\alpha_i(\sigma)(\varphi(q_1), \cdots, \varphi(q_k))_\varphi(\sigma) \leq \alpha_2(\psi(\sigma))(\varphi(q_1), \cdots, \varphi(q_k)), \quad \nu_1(\varphi(q)) \leq \nu_2(q), \quad \forall p, p_i \in Q_i, i \in [1, k], \sigma \in \Sigma_k, k \geq 0.
\]

Then we say \( M_2 \) covers the \( M_1 \), we record it as \( M_2 \leq M_1 \).

The following can be got easily:

**Proposition 3.1**. Assuming that \( M_i = (Q_i, \Sigma_i, L, \alpha_i, \nu_i)(i = 1, 2) \) is the lattice-valued tree automaton, \( i = 1, 2, 3 \). If \( M_1 \leq M_2, M_2 \leq M_3 \), then \( M_1 \leq M_3 \).

**Theorem 3.1**. Assuming that \( M_i = (Q_i, \Sigma_i, L, \alpha_i, \nu_i)(i = 1, 2) \) is the lattice-valued tree automaton, If \( \varphi : M_1 \to M_2 \) is the homomorphism, then there will be: (1) If the homomorphism is epimorphism, then \( M_2 \leq M_1 \); (2) If \( \varphi \) is the interjection, then \( M_1 \leq M_2 \).

**Proof.** \( \varphi : M_1 \to M_2 \) is the epimorphism, so there’s a full function \( \eta : Q_1 \to Q_2 \). Let \( \eta = \varphi : Q_1 \to Q_2, k, \Sigma \to \Sigma \). Obviously, \( \eta \) is the surjection. For \( \forall p, p_i \in Q_i, i \in [1, k], \sigma \in \Sigma_k, k \geq 0 \),

\[
\alpha_1(\sigma)(\varphi(q_1), \cdots, \varphi(q_k))_\varphi(\sigma) = \alpha_2(\psi(\sigma)), \quad \nu_1(\eta(p)) = \nu_2(\varphi(p)),
\]

therefore, \( M_2 \leq M_1 \).

\( \varphi : M_1 \to M_2 \) is the homomorphism, so there is the mapping \( \varphi : Q_1 \to Q_2 \). For \( \forall p, p_i \in Q_i, i \in [1, k], \sigma \in \Sigma_k, k \geq 0 \), there is

\[
\alpha_1(\sigma)(\varphi(q_1), \cdots, \varphi(q_k), \psi(\sigma)) \leq \alpha_2(\sigma)(\varphi(q_1), \cdots, \varphi(q_k)), \quad \nu_1(p) \leq \nu_2(\varphi(p)).
\]
Let \( \eta: Q_2 \rightarrow Q_1 \), \( \eta(q) = p \), \( p \in Q_1, q \in Q_2 \), if \( \varphi(p) = q \), because \( \varphi \) is the interjection, so \( p \) is the only definite, so \( \eta \) is a full partial function. Let \( 1^\omega: \Sigma \rightarrow \Sigma \), for \( \forall p, p_i \in Q_1, i \in [1,k], \sigma \in \Sigma(k), k \geq 0 \), then

\[
\alpha_{k_1}(\sigma)_{\varphi(q_1),\ldots,\varphi(q_i)}(q) \leq \alpha_{k_2}(\sigma)_{\varphi(q_1),\ldots,\varphi(q_i)}(q)
\]

\[
v_1(\eta(q)) = v_1(p) \leq v_2(\varphi(p)) = v_2(q),
\]

then \( M_1 \leq M_2 \).

**Definition 3.2.** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) (i = 1, 2) \) is the lattice-valued tree automaton, if \( \varphi \) is a full partial function from \( Q_2 \) to \( Q_1 \), and \( \psi \) is the partial function from \( \Sigma_i \) to \( \Sigma_j \), then we say \((\varphi, \psi)\) is a weak covering from \( M_2 \) to \( M_1 \), which is recorded as \( M_2 \leq_w M_1 \). For \( \forall q_1, q_2, \ldots, q_k, q \in Q_2, \sigma_i \in \Sigma_{k_i} \), it meets:

\[
\alpha_{k_i}(\sigma_i)_{\varphi(q_1),\ldots,\varphi(q_i)}(q) \leq \alpha_{k_2}(\psi(\sigma_1))_{\varphi(q_1),\ldots,\varphi(q_i)}(q),
\]

\[
v_1(\varphi(q)) \leq v_2(\varphi(p)) = v_2(q),
\]

the following can be got easily:

**Proposition 3.2.** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) \) is the lattice-valued tree automaton, \( i = 1, 2, 3 \). If \( M_1 \leq_w M_2, M_2 \leq_w M_3 \), then \( M_1 \leq_w M_3 \).

**Theorem 3.2** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) \) is the lattice-valued tree automaton, \( i = 1, 2 \). If \( (\alpha, \beta): M_1 \rightarrow M_2 \) is a homomorphism, then there are:

If this homomorphism is a strong epimorphism, and \( \alpha \) is the injection, then \( M_2 \leq_w M_1 \);

If \( \alpha \) is the injection, then \( M_1 \leq_w M_2 \).

**The Product of the Lattice-Valued Tree Automata**

**Definition 4.1.** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) (i = 1, 2) \) is the lattice-valued automaton, then the full direct product of lattice-valued automata \( M_1 \) and \( M_2 \) is \( M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, L_1 \times L_2, \alpha_1 \times \alpha_2, \nu_1 \times \nu_2) \), among them

\[
(\alpha_{k_1} \times \alpha_{k_2})(\sigma_1 \times \sigma_2)_{(p_1,q_1),\ldots,(p_i,q_i)} = \alpha_{k_1}(\sigma_1)_{p_1,\ldots,p_i} \wedge \alpha_{k_2}(\sigma_2)_{q_1,\ldots,q_i}
\]

\[
(v_1 \times v_2)(p, q) = v_1(p) \wedge v_2(q),
\]

\[
\forall (p_1,q_1), (p_2,q_2), \ldots, (p_k,q_k), (p, q) \in Q_1 \times Q_2, \sigma_1 \in \Sigma_{k_1}, \sigma_2 \in \Sigma_{k_2}, k \geq 0.
\]

**Definition 4.2.** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) (i = 1, 2) \) is the lattice-valued tree automaton, then the restricted direct product of the lattice-valued tree automata \( M_1 \) and \( M_2 \) is \( M_1 \wedge M_2 = (Q_1 \times Q_2, \Sigma_1 \wedge \Sigma_2, \alpha_1 \wedge \alpha_2, \nu_1 \wedge \nu_2) \), among them

\[
(\alpha_{k_1} \wedge \alpha_{k_2})(\sigma_1 \wedge \sigma_2)_{(p_1,q_1),\ldots,(p_i,q_i)} = \alpha_{k_1}(\sigma_1)_{p_1,\ldots,p_i} \wedge \alpha_{k_2}(\sigma_2)_{q_1,\ldots,q_i}
\]

\[
(v_1 \wedge v_2)(p, q) = v_1(p) \wedge v_2(q),
\]

\[
\forall (p_1,q_1), (p_2,q_2), \ldots, (p_k,q_k), (p, q) \in Q_1 \times Q_2, \sigma \in \Sigma_k, k \geq 0.
\]

**Definition 4.3** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) (i = 1, 2) \) is the lattice-valued tree automaton, then the cascade product of lattice-valued tree automata \( M_1 \) and \( M_2 \) is \( M_1 \circ M_2 = (Q_1 \times Q_2, \Sigma_1 \circ \Sigma_2, \alpha_1 \circ \alpha_2, \nu_1 \circ \nu_2) \), among them \( \omega: Q_1 \times X_2 \rightarrow X_1 \),

\[
(\alpha_{k_1} \circ \alpha_{k_2})(\sigma_1 \circ \sigma_2)_{(p_1,q_1),\ldots,(p_i,q_i)} = \alpha_{k_1}(\sigma_1)_{p_1,\ldots,p_i} \circ \alpha_{k_2}(\sigma_2)_{q_1,\ldots,q_i}
\]

\[
(v_1 \circ v_2)(p, q) = v_1(p) \circ v_2(q),
\]

\[
\forall (p_1,q_1), (p_2,q_2), \ldots, (p_k,q_k), (p, q) \in Q_1 \times Q_2, \sigma_1 \in \Sigma_{k_1}, \sigma_2 \in \Sigma_{k_2}, k \geq 0.
\]

**Definition 4.4.** Assuming that \( M_i = (Q_i, \Sigma_i, L_i, \alpha_i, \nu_i) (i = 1, 2) \) is the lattice-valued tree automaton, then the wreath product of lattice-valued tree automata \( M_1 \) and \( M_2 \) is
\[ M_1 \circ M_2 = (Q_1 \times Q_2, \Sigma_1^2 \times \Sigma_2, L, \alpha_1 \circ \alpha_2, v_1 \circ v_2) \], among them
\[ (\alpha_i \circ \alpha_j) (f, \sigma_2) = (\alpha_i (f(q)) (\rho (p_{-1}, p_{-2}, \ldots, p_{-n})) \wedge \alpha_j (\sigma_2) (q_{1}, q_{2}, \ldots, q_{n})), \]
\[ (v_1 \circ v_2)(p, q) = v_1(p) \wedge v_2(q), \]
\[ \forall (p_i, q_i) \in (p_2, q_2, \ldots, p_n, q_n), q_i \in Q_i, \sigma_i \in \Sigma_i, k \geq 0, \forall (f, \sigma_2) \in \Sigma_1^2 \times \Sigma_2, f : Q_2 \rightarrow \Sigma_i \]

**Theorem 4.1.** Assuming that \( M_i = (Q_i, \Sigma_i, L, \alpha_i, v_i) (i = 1, 2) \) is the lattice-valued tree automaton, then:
\[ M_1 \wedge M_2 \leq M_1 \times M_2, \text{ among them } \Sigma_i = \Sigma = \Sigma, \]
\[ M_1 \wedge \omega M_2 \leq M_1, \text{ among them } \Sigma_i = \Sigma = \Sigma, \]
\[ M_i \wedge M_2 \leq M_1 \times M_2, \]
\[ M_i \wedge M_2 \leq M_1 \times M_2. \]

**Proof.** Definition \( \varphi : Q \times Q_i \rightarrow Q \times Q_i \) is the identity mapping on \( Q \times Q_i \). Obviously, \( \varphi \) is the full partial function. Definition \( \psi : \Sigma \rightarrow \Sigma \times \Sigma \). For \( \forall \sigma \in \Sigma \), there is \( \psi (\sigma) = (\sigma, \sigma) \). \( \xi \) is a function, and
\[ (\alpha_i \wedge \alpha_j) (\sigma) (\rho (p_{-1}, p_{-2}, \ldots, p_{-n})) \wedge \alpha_j (\sigma) (q_{1}, q_{2}, \ldots, q_{n})), \]
\[ (v_1 \wedge v_2)(p, q) = v_1(p) \wedge v_2(q), \quad \text{among them,} \]
\[ \forall p, p_i \in Q_i, q, q_i \in Q_i, i = 1, 2, \ldots, n, \text{ therefore, } M_i \wedge M_2 \leq M_1 \times M_2. \]

Definition \( \varphi : Q \times Q_i \rightarrow Q \times Q_i \) is the identity mapping on \( Q \times Q_i \). Obviously, \( \varphi \) is the full partial function. Let definition \( \psi : \Sigma \rightarrow \Sigma \times \Sigma \), \( \psi (\sigma) = (f, \sigma) \). \( \forall \sigma \in \Sigma \), among them,
\[ f : Q_2 \rightarrow \Sigma, f (q_i) = \sigma = \omega (q_i, \sigma_i), \]
\[ \forall q_0 \in Q_i, \sigma_0 \in \Sigma, \psi \text{ is a function, and} \]
\[ (\alpha_{k_1} \circ \alpha_{k_2}) (f, \sigma_2) = (\alpha_{k_1} \circ \alpha_{k_2}) (f(q), (\rho (p_{-1}, p_{-2}, \ldots, p_{-n})), \wedge \alpha_{k_2} (\sigma_2) (q_{1}, q_{2}, \ldots, q_{n})), \]
\[ (v_1 \circ v_2)(p, q) = v_1(p) \wedge v_2(q), \quad \text{among them,} \]
\[ \forall p, p_i \in Q_i, q, q_i \in Q_i, i = 1, 2, \ldots, n, \text{ therefore, } M_i \wedge M_2 \leq M_1 \circ M_2. \]

Definition \( \psi : \Sigma \times \Sigma \rightarrow \Sigma \times \Sigma \) is \( f, \sigma_2 = (f(q_0), \sigma_2) \), among them, \( f : Q_2 \rightarrow \Sigma, f (q_i) = \sigma_i \), \( \forall \sigma_2 \in \Sigma \times \Sigma, q_0 \in Q_2 \), and the definition \( \varphi \) is the identity mapping on \( Q \times Q_i \), it can be easily proved that \( M_1 \circ M_2 \leq M_1 \times M_2. \)
\[\forall p, p_i \in Q, q, q_i \in Q, i = 1, 2, \ldots, n, \text{ so } M_1 \circ M_2 \leq M_1 \times M_2.\]

From (2) and (3), we can know that \(M_1 \circ M_2 \leq M_1 \times M_2.\)

The following can be easily got:

**Theorem 4.2.** Assuming that \(M_i = (Q_i, \Sigma_i, L, \alpha_i, \nu_i, \iota_i, \lambda_i) (i = 1, 2)\) is the lattice-valued tree automaton, \(i = 1, 2, 3.\) If \(M_1 \leq M_2,\) then:

\[M_1 \times M_3 \leq M_2 \times M_3, M_1 \times M_1 \leq M_3 \times M_2; \quad \text{If } \Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma, \quad \text{then} \]

\[M_1 \wedge M_3 \leq M_2 \wedge M_3, M_1 \land M_1 \leq M_3 \land M_2; \]

For any \(\omega_i : Q_i \times \Sigma_i \rightarrow \Sigma_i,\) if there is \(\omega_i : Q_i \times \Sigma_i \rightarrow \Sigma_i,\) let \(M_1 \circ \omega_2 M_3 \leq M_1 \circ \omega_2 M_3; \quad \text{If } (\varphi, \psi) \text{ is the}\]

covering of \(M_2 \text{ for } M_1,\) and \(\varphi \text{ is a surjection, then for any } \omega_i : Q_i \times \Sigma_i \rightarrow \Sigma_i,\) there is \(\omega_2 : Q_2 \times \Sigma_2 \rightarrow \Sigma_3,\) let \(M_2 \circ \omega_2 M_1 \leq M_1 \circ \omega_2 M_2; \]

\[M_1 \circ \omega_3 M_1 \leq M_2 \circ \omega_3 M_1, M_1 \circ \omega_3 M_1 \leq M_3 \circ \omega_3 M_2.\]

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**References**