Bifurcation Analysis of Discrete-Time Systems with Multiple Time-Delays

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Abstract—A unified frequency-domain approach to analyze Neimark-Sacker bifurcations and flip bifurcations of discrete-time systems with multiple time-delays in the linear feed-forward and the nonlinear term is presented. The technique relies on the harmonic balance approach (HBA) and feedback systems theory. The expressions of the bifurcation solution and the stability are derived. One example is used to verify the result obtained.

Keywords—discrete-time systems; harmonic balance approach; neimark-sacker bifurcation; flip bifurcation

I. INTRODUCTION

Bifurcation theory is a classical mathematical subject and also a very attractive research topic in many disciplines such as the physics, mechanics, chemical, biomedical and engineering systems etc. The Neimark-Sacker bifurcation occurs under assumptions that are generic, i.e. robust to perturbations. It has been observed in various applications, such as power electronics, biology, economics, neural sciences, numerical analysis, neural networks and adaptive control. For discrete-time systems, this bifurcation is sometimes called the Hopf bifurcation for map. In the continuous-time setting, the Neimark-Sacker bifurcation is often equated with the secondary Hopf bifurcation, which involves using a Poincaré map to discretize the continuous system. A flip bifurcation is also called the period-doubling bifurcation for map. It is a bifurcation in which the system switches to a new behavior with twice the period the original system. That is, there exists two points such that applying the dynamics to each of the points yields the other point. Flip bifurcations can also occur in continuous dynamical systems, namely when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. For purposes of mathematical analysis, the Neimark-Sacker bifurcation and the flip bifurcation are most conveniently considered in the discrete-time context. Most of the present literatures about the research on the Neimark-Sacker bifurcation and the flip bifurcation are based on the methods in the time domain either in the continuous-time system or in the discrete-time system, especially in the latter.

The harmonic balance approach (HBA) is a classical technique for studying nonlinear system oscillations. The main advantage of HBA is that it involves algebraic equations with less unknowns than the methods in the time domain, for problems for which low orders of approximation are sufficient to obtain an accurate solution. For this reason, in recent years,

much interest has devoted to the study of the bifurcations and chaos in nonlinear dynamical systems based on HBA. This technique has been employed in the context of continuous dynamical systems (see. [1]-[5] and the references in these text). But in the context of the discrete dynamical systems, there are relatively less contributions done in the literature. D'amico, Moiola and Paolini discuss Neimark-Sacker bifurcations and flip bifurcations of the nonlinear maps in a unified frequency-domain approach (FDA) in [6]-[8]. The method relies on HBA and feedback systems theory in the frequency domain. Zeng, Zhao, Sun and Zong analyze the Neimark-Sacker and flip bifurcations of maps with delay in the linear term via the frequency-domain method in [9].

The objective of this paper is to extends the results for maps with one delay in the linear term stated in [9] to the counterpart of the maps with multiple time-delays in the linear feed-forward and the nonlinear term. This paper is organized as follows. In section 2, HBA results are obtained for the prediction and stability analysis of Neimark-Sacker and flip bifurcations. Section 3 contains a detailed example illustrating analysis. Conclusions are contained in section 4.

II. HARMONIC BALANCE APPROACH IN NONLINEAR MAPS WITH MULTIPLE TIME-DELAYS

In this paper, we consider the following discrete-time systems with multiple time-delays:

$$\begin{aligned} x(k+1) &= A_0 x(k) + \sum_{i=1}^N A_i x(k-\tau_i) + Bg(y(k-\tau_{N+1}); \mu) \\ y(k) &= Cx(k) \end{aligned} \quad (1)$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state vector of the system, $\mu \in \mathbb{R}^s$ is the system bifurcation parameter-vector, $g(\cdot, \cdot): \mathbb{R}^m \times \mathbb{R}^s \rightarrow \mathbb{R}^l$ can be viewed as a system input, $y(k) \in \mathbb{R}^m$ is the system output, $A_i \in \mathbb{R}^{n \times n}$ ($i = 0, 1, 2, \dots, N$), $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{m \times n}$, all matrices are smooth functions of μ , $\tau_i > 0$ ($i = 1, 2, \dots, N, N+1$) are the time-delay parameters.

According to [10], there are many different but equivalent feedback representations for (1). This can be easily seen by introducing an arbitrary matrix $D \in \mathbb{R}^{l \times m}$ (which may also depend on μ) and rewriting (1) as

$$\begin{aligned} x(k+1) &= (A_0 + BDC)x(k) + \sum_{i=1}^N A_i x(k - \tau_i) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \quad (2)$$

where $u(k) = g(y(k - \tau_{N+1}); \mu) - Dy(k)$.

Taking z-transform in (2) with zero-initial conditions, yields the following representation in the frequency domain

$$Z(e) = -G(z; \mu)Z(u) \quad (3)$$

where

$$\begin{aligned} e(k) &= -y(k) = -Cx(k) \\ G(z; \mu) &= C[zI - (A_0 + BDC) - \sum_{i=1}^N A_i z^{-\tau_i}]^{-1} B \end{aligned}$$

$$u(k) = f(e(k - \tau_{N+1}); \mu) = g(-e(k - \tau_{N+1}); \mu) + De(k).$$

This representation suggests that the system (2) can be thought as the feedback interconnection, where the linear system $G(\cdot; \mu)$ and the nonlinear map $f(\cdot; \mu)$ with time-delays are defined as above.

From (3), the original problem can be solved for the variable e (contained in $Z(\cdot)$) in the frequency domain. So if $\hat{x}$ is a fixed point of (1), $\hat{e} = -C(\mu)\hat{x}$ can be considered as a fixed point in the frequency domain. Based on control theory, the fixed point $\hat{e}$ can be obtained by solving the nonlinear equation

$$\hat{e} = -G(1; \mu)f(\hat{e}; \mu), \quad (4)$$

where $G(1; \mu) = C[I - (A_0 + BDC) - \sum_{i=1}^N A_i]^{-1} B$. And the linearization of the nonlinear feedback function about $\hat{e}$ gives

$$J(\mu)z^{-\tau_{N+1}} + D = \frac{\partial g(-e; \mu)}{\partial e} \Big|_{e=\hat{e}} z^{-\tau_{N+1}} + D.$$

The characteristic equation of the linearized system is given by

$$h(\lambda, z; \mu) = \det[\lambda I - G(z; \mu)(J(\mu)z^{-\tau_{N+1}} + D)] = 0 \quad (5)$$

The critical points (ω_0, μ_0) can be deduced from $h(-1, e^{i\omega}; \mu) = 0$.

From the generalized Nyquist stability criterion, we can deduce that the crossing of an eigenvalue of (1) in the time domain through the unit circle is replaced by the crossing of one eigenvalue of $G(e^{i\omega}; \mu)(J(\mu)e^{i\omega\tau_{N+1}} + D)$, denoted as $\hat{\lambda}(e^{i\omega}; \mu)$, over the critical point $-1 + i0$ for certain frequency ω_0 in the frequency domain. Fixing $\mu = \tilde{\mu}$ and varying ω , the locus of the eigenvalue or "eigenlocus" is obtained. When there is a Neimark-Sacker bifurcation, this locus is analyzed for

$\omega_0 = \omega_H \neq 0$ excluding the strong resonance cases $e^{ir\omega_0} = 1$ for $r = 1, 2, 3, 4$. Flip bifurcations are distinguished by $\omega_0 = \omega_D = \pi$. While a unit eigenvalue, which is a necessary condition for static bifurcation, is associated with an eigenlocus that crosses the point $-1 + i0$ at $\omega_0 = 0$.

For simplicity, let $J = J(\mu)$, $G(e^{i\omega}) = G(z; \mu)|_{z=e^{i\omega}}$, $\hat{\lambda}(e^{i\omega}) = \hat{\lambda}(e^{i\omega}; \mu)$ also can be seen as a function of τ_i ($i = 1, 2, \dots, N, N+1$). Let us fix μ close to μ_0 , then the eigenlocus of $\hat{\lambda}(e^{i\omega})$ lies near the critical point $-1 + i0$. If a periodic solution exists, it can be written as the $2q$ -order approximation

$$e(k) = \hat{e} + \text{Re}\left\{\sum_{r=0}^{2q} E^r e^{ir\omega k}\right\} \quad (6)$$

where $\text{Re}\{\cdot\}$ denotes the real part of a complex number, ω is the fundamental frequency of the periodic solution, and E^r is a complex number in the r -th order harmonic of the expansion.

Expanding the nonlinear function $f(\cdot; \mu)$ with respect to $e(k)$ in Taylor series up to the $2q+1$ -order, i.e.

$$\begin{aligned} f(x+h) &= f(x; \mu) + (Df)_x h + \frac{1}{2!} (D_2 f)_x h \otimes h \\ &+ \dots + \frac{1}{j!} (D_j f)_x \otimes^j h + \mathcal{O}(\otimes^{j+1} h) \end{aligned}$$

where $\otimes$ is the standard tensor product operator, $(D_j f)_x$ can be viewed as the linear operator of the tensor-product space $\otimes^j \mathbb{R}^m$. And then replacing $e(k)$ with (6), we can obtain

$$f(e(k - \tau_{N+1}); \mu) = f(\hat{e}; \mu) + \text{Re}\left\{\sum_{r=0}^{2q} F_d^r e^{ir\omega k}\right\},$$

where F^r is a complex number in the r -th order harmonic of the expansion, which satisfies

$$F^r e^{ir\omega(k - \tau_{N+1})} = F^r e^{-ir\omega\tau_{N+1}} e^{ir\omega k} = F_d^r e^{ir\omega k}$$

Here "d" is used to indicate time-delayed quantities. Equating the output of the system with the input of the linear plant, we derive the harmonic balance equations, namely,

$$E^r = -G(e^{i\omega})F_d^r \quad r = 0, 1, \dots, 2q. \quad (7)$$

A. Neimark-Sacker Bifurcation

For analysis of Neimark-Sacker bifurcation and simplicity, let $q = 1$. By using the Taylor expansion of $f(e(k); \mu)$ at $\hat{e}$ and (7), the relation of E^r and F_d^r is the following equation

$$\begin{aligned}
E^0 &= -G(1)[(J+D)E^0 + \frac{1}{4}D_2g(E^1 \otimes \bar{E}^1)] + \mathcal{O}(|E^1|^4) \\
E^1 &= -G(e^{i\omega})[(Je^{-i\omega T_{N+1}} + D)E^1 \\
&\quad + 1/2 \cdot D_2g(2E^0 \otimes E^1 e^{-i\omega T_{N+1}} + \bar{E}^1 \otimes E^2) e^{-i\omega T_{N+1}} \\
&\quad + 1/6 \cdot D_3g(3/4E^1 \otimes E^1 \otimes \bar{E}^1 e^{-i\omega T_{N+1}})] + \mathcal{O}(|E^1|^4) \\
E^2 &= -G(e^{i2\omega})[(Je^{-i2\omega T_{N+1}} + D)E^2 \\
&\quad + 1/4D_2g(E^1 \otimes E^1 e^{-i2\omega T_{N+1}})] + \mathcal{O}(|E^1|^4)
\end{aligned} \quad (8)$$

where $D_jg = D_jg(e; \mu)|_{e=e^i}$. Similar as in [10], these equations are solved in terms of $E^1 = \theta v$, where v is the normalized right eigenvector associated with the eigenvalue $\hat{\lambda}(e^{i\omega})$ (the eigenvalue λ satisfying (5) that is closest to the critical point $-1+i \cdot 0$), and θ is a measure of the amplitude of the periodic solution. The other complex vectors are given by $E^0 = V_{0H}\theta^2$ and $E^2 = V_{2H}\theta^2$ with

$$\begin{aligned}
V_{0H} &= -1/4H(1; \mu)Q\bar{v} \\
V_{2H} &= -1/4H(e^{i2\omega}; \mu)Qve^{-i2\omega T_{N+1}}
\end{aligned}$$

where $H(z; \mu) = [I + G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)]^{-1}G(z; \mu)$, $Q = (D_2g)v$ can be computed as in [9].

So, for the periodic solution (1) to exist, it is necessary that

$$\hat{\lambda}(e^{i\omega}; \mu) = -1 + \xi_{Hd}\theta^2 + \mathcal{O}(\theta^3) \quad (9)$$

with

$$\xi_{Hd} = -\frac{s^T G(e^{i\omega})p_{Hd}}{s^T v}$$

Where $p_{Hd}(\omega, v, \tau) = [QV_{0H} + 1/2QV_{2H} + 1/8L\bar{v}]e^{-i\omega T_{N+1}}$, s is the normalized left eigenvector associated with the eigenvalue $\hat{\lambda}(e^{i\omega})$, $L = (D_3g)v^2$.

Similar as in [9], we can prove that the stability of the emerging periodic solution can be determined by the stability index σ_{Hd} , i.e.

$$\sigma_{Hd} = \text{Re}\left\{\frac{s^T G(e^{i\omega})p_{Hd}e^{-i\omega}}{s^T \frac{\partial}{\partial z}[G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)]|_{z=e^{i\omega}}}v\right\} \quad (10)$$

If $\sigma_{Hd} > 0$ ($\sigma_{Hd} < 0$), there experiences the supercritical (subcritical) Neimark-Sacker bifurcation and the emerging invariant circle is stable (unstable). Case with $\sigma_{Hd} = 0$ has been discussed in [8, 11].

The following is the proof of the stability index σ_{Hd} . Supposing that the matrix $G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)$ has an

eigenvalue $\hat{\lambda}(z_0; \mu_0) = -1 + i \cdot 0$, it is easy to verify that the matrix $G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)$ has the same eigenvalue. Moreover, at the critical value $\mu = \mu_0, \omega = \omega_0$, the following hold: $z_0 = e^{i\omega_0}, \bar{z}_0 = e^{-i\omega_0}$, which are exactly the two corresponding complex eigenvalues of (1) of the discrete-time version of the HBA for a given value $\mu = \mu_0$ according with the Neimark-Sacker bifurcation.

Without lose of generality, supposing that when $\mu > \mu_0$, the pair of complex conjugate eigenvalues of (9) crosses the unit circle, i.e. $z = \rho e^{i\omega}$ ($\rho > 1$). What's more, considering that under this condition, the analysis in the frequency domain guarantees the existence of a periodic solution with an approximate frequency $\hat{\omega}$ and a small amplitude θ . Then we can write

$$\begin{aligned}
G(e^{i\hat{\omega}}; \mu)(J(\mu)e^{-i\hat{\omega}T_{N+1}} + D) &= G(z; \mu)(J(\mu)z^{-T_{N+1}} + D) \\
&+ (e^{i\hat{\omega}} - z) \frac{d}{dz}[G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)] \\
&+ \frac{1}{2!}(e^{i\hat{\omega}} - z)^2 \frac{d^2}{dz^2}[G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)] + \dots
\end{aligned} \quad (11)$$

Substituting (7) into the second equation of (8), and letting $E^1 = \theta(v + w)$, then we can derive

$$\begin{aligned}
&[G(e^{i\hat{\omega}}; \mu)(J(\mu)e^{-i\hat{\omega}T_{N+1}} + D) + I](v + w) \\
&= -G(e^{i\hat{\omega}}; \mu)p_{Hd}(\hat{\omega}, v + w, \tau)\theta^2 + \mathcal{O}(\theta^3)
\end{aligned} \quad (12)$$

First multiplying both sides s^T , then substituting (11) into the equation and considering $w = \mathcal{O}(\theta^2)$, so we can get

$$\begin{aligned}
&s^T [(e^{i\hat{\omega}} - z) \frac{d}{dz}[G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)] + \dots]v \\
&= -s^T G(e^{i\hat{\omega}}; \mu)p_{Hd}(\hat{\omega}, v, \tau)\theta^2 + \mathcal{O}(\theta^3)
\end{aligned} \quad (13)$$

Observing that $e^{i\hat{\omega}}$ is in the small neighborhood of $e^{i\omega}$, So

$$z - e^{i\hat{\omega}} = \gamma\theta^2 + \mathcal{O}(\theta^3),$$

where

$$\gamma = \frac{s^T G(e^{i\omega}; \mu)p_{Hd}(\omega, v, \tau)}{s^T \frac{d}{dz}[G(z; \mu)(J(\mu)z^{-T_{N+1}} + D)]|_{z=e^{i\omega}}}v$$

and

$$\rho - \text{Re}\{e^{i(\hat{\omega}-\omega)}\} = \text{Re}\{\gamma e^{-i\omega}\}\theta^2 + \mathcal{O}(\theta^3).$$

So for a small amplitude $\theta^2 > 0$, the necessary condition for which the above equation has a solution is that $\sigma_{Hd} = \text{Re}\{\gamma e^{-i\omega}\}$ and $\rho - \text{Re}\{e^{i(\hat{\omega}-\omega)}\}$ have the same sign. Here σ_{Hd} is the stability index. If $\rho > 1$, $\sigma_{Hd} > 0$. What's more, the fixed point become

unstable and the stable closed invariant circle is bifurcating from the fixed point when μ is near to μ_0 and $\mu > \mu_0$. In the same way, we can prove the following results: (i) If $\rho < 1$, $\sigma_{Hd} < 0$. (ii) The unstable closed invariant circle is bifurcating from the fixed point when μ is near to μ_0 and $\mu < \mu_0$.

B. Flip Bifurcation

As for analysis of the flip bifurcation, let $q=1$ and $\omega = \omega_D = \pi$. Then the period solution emerging from the flip bifurcation can be written as

$$e(k) = \hat{e} + E^0 + E^1 e^{i\pi k}, \quad (14)$$

and the harmonic balance equations can be derived as

$$E^0 = -G(1; \mu) F_d^0, \quad E^1 = -G(e^{i\pi}; \mu) F_d^1 \quad (15)$$

where

$$\begin{aligned} F_d^0 &= (J + D)E^0 + \frac{1}{2}D_2g(E^1 \otimes E^1 e^{-i2\pi T_{N+1}}) + \mathcal{O}(|E^1|^4) \\ F_d^1 &= (Je^{-i\pi T_{N+1}} + D)E^1 + D_2g(E^0 \otimes E^1 e^{-i\pi T_{N+1}}) \\ &\quad + 1/6D_3g(E^1 \otimes E^1 \otimes E^1 e^{-i3\pi T_{N+1}}) + \mathcal{O}(|E^1|^4) \end{aligned} \quad (16)$$

Analyzing the flip bifurcation as above section II-A, then the eigenvalue $\hat{\lambda}(e^{i\omega})$ satisfies

$$\hat{\lambda}(-1; \mu) = -1 + \xi_D \theta^2 + \mathcal{O}(\theta^3) \quad (17)$$

and the stability of the emerging periodic solution can be determined by the stability index σ_{Dd}

$$\sigma_{Dd} = \text{Re}\left\{-\frac{s^T G(-1; \mu_0) p_{Dd}(v, \tau)}{s^T \frac{\partial}{\partial z} [G(z; \mu_0)(J(\mu_0)z^{-T_{N+1}} + D)]|_{z=-1}} v\right\} \quad (18)$$

where

$$\begin{aligned} p_{Dd}(v, \tau) &= (QV_{0D} + 1/6Lve^{-i2\pi T_{N+1}})e^{-i\pi T_{N+1}} \\ \xi_D &= -\frac{s^T G(-1; \mu) p_{Dd}(v, \tau)}{s^T v} \end{aligned}$$

(π, μ_0) satisfies $h(-1, e^{i\pi}; \mu_0) = 0$. If $\sigma_{Dd} > 0$ ($\sigma_{Dd} < 0$), there experiences the supercritical (subcritical) flip bifurcation and the emerging invariant circle is stable (unstable). Case with $\sigma_{Dd} = 0$ has been discussed by using normal form theory and singular theory in [12].

III. EXAMPLE

Consider the difference equation with two time-delays

$$x(k+1) = \mu x(k) + ax(k - \tau_1) + bx(k - \tau_2) - \mu x^2(k - \tau_3) \quad (19)$$

where τ_i ($i=1,2,3$) are integer time-delays, $0 < a, b < 1$. Here we only consider $b=2a$, $\tau_1=1$, $\tau_2=2$ for simplicity. At the fixed point $\hat{x} = 1 + \frac{a+b-1}{\mu} = 1 + \frac{3a-1}{\mu}$ of (19), firstly considering the Neimark-Sacker bifurcation for $1/3 < a < 1/2$. Selecting $B=C=1, D=0$, then

$$\begin{aligned} A_0 &= \mu, A_1 = a, A_2 = b = 2a, \\ f(e(k - \tau_3); \mu) &= -\mu e^2(k - \tau_3), \\ G(z; \mu) &= C(zI - A_0 - A_1 z^{-1} - A_2 z^{-2})^{-1} B \\ &= \frac{1}{z - \mu - \frac{a}{z} - \frac{2a}{z^2}} \end{aligned}$$

where $e(k) = -Cx(k) = -x(k)$, the fixed point in the frequency domain is $\hat{e} = \frac{1-3a}{\mu} - 1$. The open-loop system matrix is

$$G(e^{i\omega}; \mu) J(\mu) e^{-i\omega T_3} = \frac{2\mu - 2 + 6a}{e^{i\omega} - \mu - ae^{-i\omega} - 2ae^{-i2\omega}} e^{-i\omega T_3}$$

with its eigenvalue

$$\lambda(e^{i\omega}; \mu) = G(e^{i\omega}; \mu) J(\mu) e^{-i\omega T_3}.$$

Let $\hat{\lambda}(\omega) = \lambda(e^{i\omega}; \mu)$, then the associate left and right eigenvector are $s^T = v = 1$.

The matrices Q and L are

$$Q = -2\mu \quad L = 0$$

and the linearized close-loop function is

$$H(z) = \frac{z^2}{z^3 + (\mu - 2 + 6a)z^2 - az - 2a}.$$

Then

$$V_{0H} = \frac{\mu}{2(\mu - 1 + 3a)}$$

$$V_{2H} = \frac{\mu e^{i4\omega} e^{-i2\omega T_3}}{2[e^{i6\omega} + (\mu - 2 + 6a)e^{i4\omega} - ae^{i2\omega} - 2a]}$$

and

$$\xi_{Hd} = -\frac{e^{i2\omega}}{e^{i3\omega} - \mu e^{i2\omega} - ae^{i\omega} - 2a} p_{Hd}$$

with

$$\begin{aligned} p_{Hd} &= e^{-i\omega T_3} \left\{ -\frac{\mu^2}{\mu - 1 + 3a} \right. \\ &\quad \left. - \frac{\mu^2 e^{i4\omega}}{2[e^{i6\omega} + (\mu - 2 + 6a)e^{i4\omega} - ae^{i2\omega} - 2a]} \right\}. \end{aligned}$$

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Since

$$\hat{\lambda}(\omega) = -1 + \xi_{Hd} \theta^2$$

can be thought as the intersection between the locus of $\hat{\lambda}(\omega)$ and a half-line starting at $-1+i0$ in the direction defined by $\xi_{Hd}(\omega)$. Denote the intersection point by $q(\hat{\omega}, \mu) = \hat{\lambda}(\hat{\omega})$, which is attained by using graphical method, then we can compute the approximate frequency $\hat{\omega}$ and its amplitude $\theta = [1 + q / \xi_{Hd}]^{1/2}$. So the approximate expression of the invariant circle emerging from Hopf bifurcation is as follows

$$x(k) = -e(k) = -\hat{e} - \text{Re}\{\theta^2 V_{0H} + \theta v e^{ik\hat{\omega}} + \theta^2 V_{2H} e^{i2k\hat{\omega}}\}.$$

The stability of the emerging periodic solution is decided by

$$\sigma_{Hd} = \text{Re}\left\{\frac{e^{i3\hat{\omega}} - \mu e^{i2\hat{\omega}} - a e^{i\hat{\omega}} - 2a}{2(\mu - 1 + 3a)(-e^{i3\hat{\omega}} - a e^{i\hat{\omega}} - 4a)} p_{Hd}\right\}$$

Specifically, for $\tau_3 = 0$, the critical value $\mu_0 = \frac{1}{2a} + \frac{5}{2} - 8a$,

$\cos(\omega_0) = -\frac{1}{4} - \frac{1}{4a}$ for $a > 1/3$, then $\sigma_{Hd} > 0$ for $0.3632 < a < 0.5$, so the closed invariant circle bifurcating from the fixed point is stable. We also have the closed invariant circle is unstable since $\sigma_{Hd} < 0$ when $1/3 < a < 0.3632$.

Assuming, for instance, that $\mu = 0.56$ for $a = 0.4$ ($\mu_0 = 0.55$), the intersection point is $q = -1.012 - 0.001249i$ at $\hat{\omega} = 3.7$, the locus of $\hat{\lambda}(\omega)$ and the half-line starting at $-1+i0$ in the direction defined by $\xi_{Hd}(\hat{\omega})$. The locus intersects the half-line verifying the existence of an invariant cycle.

In the same way, we can consider the flip bifurcation (19) at the point $\hat{x} = 1 + \frac{3a-1}{\mu}$ for $\mu = \mu_0 = 3 - 5a$ ($a < 1/3$). By

computing $\sigma_{Dd} = \frac{\mu^2(1+\mu+a)}{(\mu-1+3a)^2(1-3a)} > 0$. For instance,

assuming that $a = 0.1$, then $\mu_0 = 2.5$, $\omega_0 = \pi$ and the stable 2-periodic solutions are bifurcating from the fixed point

$$\hat{x} = 1 + \frac{3a-1}{\mu}.$$

IV. CONCLUSION

A frequency domain approach to detect Neimark-Sacker and flip bifurcations in the unified formulation for discrete-time systems with multiple time-delays has been discussed by using harmonic balance approach. And the expressions that predict the emerging periodic solution and determine the stability of the periodic solution have been derived. An extension to deal with bifurcation control is very attractive in the future since FDA for maps provides a useful graphical interpretation.