Solving the kinetic equations with geometric nonlinearity considered

Duan Jin^{1,a}, Li Yun-gui¹

¹ China State Construction Technical Center, Beijing, 101300, PR China

^a duanjin78@126.com

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Abstract: This paper presents a new time integration method to solve the kinetic equations with geometric nonlinearity considered, by combining the Newmark integration method and Newton-Raphson iteration method. In this method, the Newmark method is used for the time history integration and the Newton-Raphson iteration method is adopted to solve the nonlinear equation at current time step. The solving procedure of nonlinear dynamic equations is divided into two procedures, i.e. prediction and correction. Finally, a numerical example of a clamped beam subjected to a concentrated step load is presented to verify the validity and accuracy of the present method.

Introduction

Bathe [1] has pointed out that solving nonlinear dynamic equations is indeed the combination of solving linear dynamic equations and solving nonlinear static equations. For the solution of nonlinear static equation, Newton-Raphson iteration method is possibly the most classical method. It is suitable for the positive definite stiffness matrix and unsuitable for the negative definite stiffness matrix. Fortunately the majorities of finite element stiffness matrix are positive definite. The most marked feature of Newton-Raphson iteration is that there is a defined constant load increment according to every solution increment. In terms of the solution of linear dynamic equation, it can be classified to two kinds of methods, i.e. direct integrant method and mode superposition method. The direct integrant method can be furtherly divided into explicit method and implicit method according its time integrant scheme. For the explicit method, the displacement at next time would be calculated by satisfying the dynamic government equation at current time. While for the implicit method, the displacement at next time would be calculated by satisfying the dynamic government equation at next time. The explicit method is conditional stable while the implicit method is non-conditional stable [2]. The most famous explicit method may be central difference method and the corresponding implicit method may be Newmark method [3].

In this paper, by combining the Newmark integration method and Newton-Raphson iteration method, a new time integration method is presented to solve the kinetic equations with geometric nonlinearity considered. In this method, the Newmark method is used for the time history integration and the Newton-Raphson iteration method is adopted to solve the nonlinear equation at current time step. The solving procedure of nonlinear dynamic equations is divided into two procedures, i.e. prediction and correction. Finally, a numerical example of a clamped beam subjected to a concentrated step load is presented to verify the validity and accuracy of the present method.

Solution of the dynamics equation considering geometric nonlinearity

The dynamic government equation can be described usually as follow ^[1]:

$$\mathbf{M}^{t+\Delta t} \ddot{\mathbf{a}} + \mathbf{D}^{t+\Delta t} \dot{\mathbf{a}} + \mathbf{K} \left({}^t\mathbf{a} \right) \Delta \mathbf{a} = {}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{F} \left({}^t\mathbf{a} \right) \quad (1)$$

The above equation can be transformed to the following form using the Newton-Raphson iteration method [1]:

$${}^{t+\Delta t}\mathbf{K}^{(k-1)} \Delta \mathbf{a}^{(k)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} - \mathbf{M}^{t+\Delta t} \ddot{\mathbf{a}} - \mathbf{D}^{t+\Delta t} \dot{\mathbf{a}} \quad (2)$$

$${}^{t+\Delta t}\mathbf{a}^{(k)} = {}^{t+\Delta t}\mathbf{a}^{(k-1)} + \Delta \mathbf{a}^{(k)} \quad (3)$$

where

$${}^{t+\Delta t}\mathbf{a}^{(0)} = {}^t\mathbf{a} \quad (4)$$

$${}^{t+\Delta t}\mathbf{K}^{(0)} = {}^t\mathbf{K}; \quad {}^{t+\Delta t}\mathbf{F}^{(0)} = {}^t\mathbf{F} \quad (5)$$

For the Newmark integration scheme, the following hypothesis would be adopted for the time step t to $t + \Delta t$:

$$\begin{cases} {}^{t+\Delta t}\mathbf{a} = {}^t\mathbf{a} + {}^t\ddot{\mathbf{a}}\Delta t + \left(\left(\frac{1}{2} - a \right) {}^t\ddot{\mathbf{a}} + a {}^{t+\Delta t}\ddot{\mathbf{a}} \right) \Delta t^2 \\ {}^{t+\Delta t}\dot{\mathbf{a}} = {}^t\dot{\mathbf{a}} + \left((1-d) {}^t\dot{\mathbf{a}} + d {}^{t+\Delta t}\dot{\mathbf{a}} \right) \Delta t \end{cases} \quad (6)$$

Where a and d denotes the Newmark parameters respectively and in order to satisfy the stability condition of Newmark method, a and d must satisfy the following equation:

$$d \geq 0.5 \quad \text{and} \quad a \geq 0.25(0.5 + d)^2 \quad (7)$$

Substitute Equation (3) into ${}^{t+\Delta t}\mathbf{a}$, Equation (6) can be transformed to the following one:

$$\begin{cases} {}^{t+\Delta t}\mathbf{a}^{(k-1)} + \Delta \mathbf{a}^{(k)} = {}^t\mathbf{a} + {}^t\ddot{\mathbf{a}}\Delta t + \left(\left(\frac{1}{2} - a \right) {}^t\ddot{\mathbf{a}} + a {}^{t+\Delta t}\ddot{\mathbf{a}} \right) \Delta t^2 \\ {}^{t+\Delta t}\dot{\mathbf{a}} = {}^t\dot{\mathbf{a}} + \left((1-d) {}^t\dot{\mathbf{a}} + d {}^{t+\Delta t}\dot{\mathbf{a}} \right) \Delta t \end{cases} \quad (8)$$

Then

$$\begin{cases} {}^{t+\Delta t}\ddot{\mathbf{a}} = c_0 \left({}^{t+\Delta t}\mathbf{a}^{(k-1)} - {}^t\mathbf{a} + \Delta \mathbf{a}^{(k)} \right) - c_2 {}^t\ddot{\mathbf{a}} - c_3 {}^t\dot{\mathbf{a}} \\ {}^{t+\Delta t}\dot{\mathbf{a}} = c_1 \left({}^{t+\Delta t}\mathbf{a}^{(k-1)} - {}^t\mathbf{a} + \Delta \mathbf{a}^{(k)} \right) - c_4 {}^t\dot{\mathbf{a}} - c_5 {}^t\ddot{\mathbf{a}} \end{cases} \quad (9)$$

Where $c_0 \sim c_5$ denotes the Newmark integration constants, see the following equation:

$$c_0 = \frac{1}{a\Delta t^2}, \quad c_1 = \frac{d}{a\Delta t}, \quad c_2 = \frac{1}{a\Delta t}, \quad c_3 = \frac{1}{2a} - 1, \quad c_4 = \frac{d}{a} - 1, \quad c_5 = \frac{\Delta t}{2} \left(\frac{d}{a} - 2 \right) \quad (10)$$

Substitute Equation (9) into Equation (2), then:

$${}^t\hat{\mathbf{K}}\Delta \mathbf{a}^{(k)} = \hat{\mathbf{R}} \quad (11)$$

where

$${}^t\hat{\mathbf{K}} = {}^{t+\Delta t}\mathbf{K}^{(k-1)} + c_0\mathbf{M} + c_0\mathbf{D} \quad (12)$$

$$\begin{aligned} \hat{\mathbf{R}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(k-1)} + \mathbf{M} \left(c_0 \left({}^t\mathbf{a} - {}^{t+\Delta t}\mathbf{a}^{(k-1)} \right) + c_2 {}^t\ddot{\mathbf{x}} + c_3 {}^t\ddot{\mathbf{y}} \right) \\ + \mathbf{D} \left(c_1 \left({}^t\mathbf{a} - {}^{t+\Delta t}\mathbf{a}^{(k-1)} \right) + c_4 {}^t\ddot{\mathbf{x}} + c_5 {}^t\ddot{\mathbf{y}} \right) \end{aligned} \quad (13)$$

In Equation (11), the current iteration would cease and enter next time step when $\|\Delta\mathbf{a}^{(k)}\| \leq \text{TOL}$. Here TOL denotes a defined tolerance.

From Equation (11-13) and (3-5), it is obtained that the solving procedure of nonlinear dynamic equations could be roughly divided into two procedures, i.e. prediction and correction [3], see the following for details.

The first procedure: prediction ($k = 1$)

- (1) Calculating Newmark integration constants according to Equation (10);
- (2) Calculating the initial values of node displacement and force according to Equation (4-5);
- (3) Calculating equivalent stiffness matrix ${}^t\hat{\mathbf{K}}$ and load vector $\hat{\mathbf{R}}$ according to Equation (12-13);
- (4) Solving $\Delta\mathbf{a}^{(1)}$ by Equation (11);
- (5) Calculating ${}^{t+\Delta t}\mathbf{a}^{(1)}$ according to Equation (3);
- (6) If $\|\Delta\mathbf{a}^{(1)}\| \leq \text{TOL}$, terminating the current iteration, calculating nodes velocity and acceleration according to Equation (9)m and entering the next time step; otherwise entering the second procedure as following.

The second procedure: correction ($k = k + 1$)

- (1) Calculating stiffness matrix ${}^t\hat{\mathbf{K}}$ and unbalanced force $\hat{\mathbf{R}}$ according to Equation (12) and (13) respectively;
- (2) Solving $\Delta\mathbf{a}^{(k)}$ by Equation (11);
- (3) Calculating ${}^{t+\Delta t}\mathbf{a}^{(k)}$ according to Equation (3);
- (4) If $\|\Delta\mathbf{a}^{(k)}\| \leq \text{TOL}$, terminating the current iteration, calculating nodes velocity and acceleration according to Equation (9)m and entering the next time step; otherwise looping the second procedure as above.

Numerical example

Figure 1 shows a clamped beam subjected to a concentrated step load. The central deflection response is shown in Figure 2, where the results from nonlinear analysis with time step $\Delta t = 25 \times 10^{-6} \text{ s}$ are measured by the left scale while the results from linear analysis with time step $\Delta t = 100 \times 10^{-6} \text{ s}$ are measured by the right scale. The validity of the proposed method is further confirmed from the comparison with the numerical results of Yang and Saigal [5], Mondkar and Powell [6]. Again, it observes that the amplitude and period of linear analysis are much larger than those of nonlinear analysis, because the nonlinear stiffness of this structure is much larger than the linear stiffness.

Figure 1 A clamped beam subjected to a concentrated step load

Figure 2 Large deflection dynamic responses of clamped beam subjected to a concentrated step load

Summary

In this paper, by combining the Newmark integration method and Newton-Raphson iteration method, a new time integration method is presented to solve the kinetic equations with geometric nonlinearity considered. In this method, the Newmark method is used for the time history integration and the Newton-Raphson iteration method is adopted to solve the nonlinear equation at current time step. The solving of nonlinear dynamic equations is divided into two procedures, i.e. prediction and correction. Finally, a numerical example of a clamped beam subjected to a concentrated step load is presented to verify the validity and accuracy of the present method.

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