

Several Notations of Gronwall Inequality of Vector Functions

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Abstract In this paper, we first consider two inequalities of vector functions, then combine the vector function inequality with Gronwall's inequality to obtain another two vector function's inequalities. At last, using them to prove the maximum principle for one kind of optimal control problem.

1 Introduction and derivation of vector functions' inequality

Gronwall inequalities show the relationship among functions and their derivatives and integration [1], and are widely used to estimate functions' value in function spaces such as L^p space so on. They are very important for embedding theories of Sobolev space [2,3], also are key tools for research on the solution of PDE [4] and some problems of optimal control of differential equation [5,6,7]. For convenience, we first list the two lemmas bellowing:

Lemma.1 [8] Let $y(t)$ be a continuously differential vector value function on the interval $[t_0, t_1]$, then $\frac{d}{dt}\|y(t)\| \leq \left\| \frac{d}{dt} y(t) \right\|$, for all $t \in [t_0, t_1]$, and $y(t) \neq 0$.

Lemma.2. [9] Let $y(t)$ be continuous vector valued function on $[t_0, t_1]$. If $y(t)$ satisfies $\|y(t)\| \leq a\varepsilon + \beta \int_{t_1}^t \|y(\delta)\| d\delta, \forall t \in [t_1, t_2]$ for given positive constant a, β, ε , then $\|y(t)\| \leq a\varepsilon e^{\beta(t-t_1)}, \forall t \in [t_1, t_2]$.

Then show a vector function inequality from Gronwall inequality.

Proposition.3. Let $y(t)$ be continuously differential vector-valued function on the interval $[t_0, t_1]$, and $y_i'(t) \leq \phi_i(t)y_i(t) + \psi_i(t), i = 1, 2, \dots, n$, then

$$\|y(t)\| \leq e^{\int_{t_0}^t \|\phi(s)\| ds} [\|y(t_0)\| + \int_{t_0}^t \|\psi(s)\| ds], \quad (1)$$

Where $y(t) = (y_1(t), \dots, y_n(t))$, $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$, $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))$

and $\|y(t)\|, \|\phi(t)\|, \|\psi(t)\|$ are the Euclidean norms correspondingly.

Proof. From lemma.1., Minkowski inequality and Hölder inequality, we can obtain

$$\|y(t)\|' \leq \|\phi(t)\| \|y(t)\| + \|\psi(t)\|.$$

Hence through the differential form Gronwall inequality, we obtain

$$\|y(t)\| \leq e^{\int_{t_0}^t \|\phi(s)\| ds} [\|y(t_0)\| + \int_{t_0}^t \|\psi(s)\| ds]. \quad (2)$$

Proposition.4. Let $y(t)$ be continuous vector function on the interval $[t_0, t_1]$

If $y(t)$ satisfies $\|y(t)\| \leq a\varepsilon + \beta \int_{t_1}^t \|y(\delta)\| d\delta, \forall t \in [t_1, t_2]$ for given positive constant a, β, ε . Then $\|y(t)\| \leq a\varepsilon [1 + \beta(t-t_0)e^{\beta(t-t_0)}]$.

The proof is trivial with integral form Gronwell inequality.

2 An Example of the Inequalities' Application

Example. Consider the optimal control problem which system is stated by

$\dot{x} = f(x, u)$, $x(t_0) = x_0$, where $x \in R^n$, control variable $u \in R^r$, the objective set is $x(t_f) \in R^n$ and t_f is fixed. $f : R^n \times R^r \rightarrow R^n$ is C^1 , $U_{ad} = \{u(t) | u(t) \in U_r \subseteq R^r\}$,

where $u(t)$ is piece-wise continuous functions and $U_r \in R^r$ is bounded closed set, $x(t)$ is the state variable corresponding to the control variable $u(\cdot)$ and its initial state satisfies $x(t_0) = x_0$, set the cost functional as following

$$J[u(\cdot)] = K(x(t_f)) + \int_{t_0}^{t_f} L(x, u) dt. \quad (3)$$

To ensure the existence of the solution of the optimal control problem, we assume that

(i) $f(x, u)$, $L(x, u)$, $K(x)$ are continuous for their variable and are continuously differential to the variable x ; (ii) $f(x, u)$, $\frac{\partial f(x, u)}{\partial x}$, $\frac{\partial L(x, u)}{\partial x}$ all are bounded.

Since $f(\bar{x}, u) - f(\hat{x}, u) = \int_{\hat{x}}^{\bar{x}} \frac{\partial f(x, u)}{\partial x} dx$, there are two positive constants a, b which are independent to x, u such that $\|f(x, u)\| \leq \frac{a}{2}$, $\forall u \in U, x \in R^n$, $\|f(\bar{x}, u) - f(\hat{x}, u)\| \leq b \|\bar{x} - \hat{x}\|$, where $\|\cdot\|$ is the norm of Euclidean vector.

Proposition5 if $u^*(t)$ is the optimal control of the above problem, the necessary condition is that $H(x^*(t), u^*(t), \psi(t)) = \max_{u \in U_r} H(x^*(t), u(t), \psi(t))$.

Proof. Let $\psi(t) \in R^n$, then problem (4) can be transformed to the problem

$$J[u(\cdot)] = K(x(t_f)) + \psi^T(t_f)x(t_f) - \psi^T(0)x_0 - \int_{t_0}^{t_f} [\psi^T(t)x(t) + H(x(t), u(t), \psi(t))] dt \quad (4)$$

without constrain condition and where

$H(x, u, \psi) = -L(x, u) + \psi^T f(x, u)$ is the Hamilton function, assume $(u^*(t), x^*(t))$ be the optimal pairs of the optimal control problem. Given

$$u(t) = u^*(t) + \Delta u(t), t \in [t_0, t_f] \quad (5)$$

to ensure $u(t) \in U_{[t_0, t_f]}$, then $\Delta u(t)$ must satisfies two conditions : $\Delta u(t)$ define on $[t_0, t_f]$, and are continuous vector functions, $u(t) = u^*(t) + \Delta u(t) \in U_r, t \in [t_0, t_f]$.

And $x(t) = x^*(t) + \Delta x(t), t \in [t_0, t_f]$ is the solution of optimal control problem corresponding to $u(t)$ above. It is trivial that $\Delta x(t)$ satisfies

$$\begin{cases} \frac{d}{dt} \Delta x(t) = f(x^*(t) + \Delta x(t), u^*(t) + \Delta u(t)) - f(x^*(t), u^*(t)), \\ \Delta x(t_0) = 0. \end{cases} \quad (6)$$

Then we obtain

$$\Delta J[u^*(\cdot)] \triangleq J[u(\cdot)] - J[u^*(\cdot)] = K(x^*(t_f) + \Delta x^*(t_f)) - K(x^*(t_f)) + \psi^T(t_f) \Delta x(t_f)$$

$$-\int_{t_0}^{t_f} \dot{\psi}(t) \Delta x(t) dt - \int_{t_0}^{t_f} \left[H(x^*(t) + \Delta x(t), u^*(t) + \Delta u(t), \psi(t)) - H(x^*(t), u^*(t), \psi(t)) \right] dt.$$

And according to the assumptions of $f(x, u), L(x, u), K(x)$, we get

$$K(x^*(t_f) + \Delta x(t_f)) - K(x^*(t_f)) = \frac{\partial K(x^*(t_f))}{\partial x} \Delta x(t_f) + o(\|\Delta x(t_f)\|), \quad (7)$$

$$\begin{aligned} \Delta H(x^*(t), u^*(t), \psi(t)) &= H(x^*(t) + \Delta x(t), u^*(t) + \Delta u(t), \psi(t)) - H(x^*(t), u^*(t), \psi(t)) \\ &= \left\{ \frac{\partial F(x^*(t), u^*(t) + \Delta u(t), \psi(t))}{\partial x} + \frac{\partial H(x^*(t), u^*(t), \psi(t))}{\partial x} \right\} \Delta x(t) \\ &\quad + o(\|\Delta x(t)\|) + F(x^*(t), u^*(t) + \Delta u(t), \psi(t)). \end{aligned} \quad (8)$$

where $F(x^*(t), u^*(t) + \Delta u(t), u^*(t), \psi(t)) = H(x^*(t), u^*(t) + \Delta u(t), \psi(t)) - H(x^*(t), u^*(t), \psi(t))$ then

$$\begin{aligned} \Delta J[u^*(\cdot)] &= \left[\frac{\partial K(x^*(t_f))}{\partial x} + \psi^T(t_f) \right] \Delta x(t_f) - \int_{t_0}^{t_f} \left[\dot{\psi}(t) + \frac{\partial H}{\partial x} \Big|_* \right] \Delta x(t) dt \\ &\quad - \int_{t_0}^{t_f} \frac{\partial F(x^*(t), u^*(t) + \Delta u(t), u^*(t), \psi(t))}{\partial x} \Delta x(t) dt \\ &\quad - \int_{t_0}^{t_f} F(x^*(t), u^*(t) + \Delta u(t), u^*(t), \psi(t)) dt + \int_{t_0}^{t_f} o(\|\Delta x(t)\|) dt + o(\|\Delta x(t_f)\|), \end{aligned}$$

where

$$\frac{\partial H}{\partial x} \Big|_* = \frac{\partial H(x^*(t), u^*(t), \psi(t))}{\partial x}.$$

As we know that $\psi(t)$ satisfies the condition below:

$$\begin{cases} \dot{\psi}^T(t) = -\frac{\partial H(x^*(t), u^*(t), \psi(t))}{\partial x} \\ \psi^T(t_f) = -\frac{\partial K(x^*(t_f))}{\partial x} \end{cases}. \quad (9)$$

Then

$$\begin{aligned} \Delta J[u^*(\cdot)] &= -\int_{t_0}^{t_f} \frac{\partial F(x^*(t), u^*(t) + \Delta u(t), u^*(t), \psi(t))}{\partial x} \Delta x(t) dt \\ &\quad - \int_{t_0}^{t_f} F(x^*(t), u^*(t) + \Delta u(t), u^*(t), \psi(t)) dt + \int_{t_0}^{t_f} o(\|\Delta x(t)\|) dt + o(\|\Delta x(t_f)\|) dt. \end{aligned} \quad (10)$$

As we know that for any $\Delta u(t)$ of $u^*(t)$, (9) hold. So when we chose a special

$$u(t) = \begin{cases} u^*(t), & t \in [t_0, \bar{t}) \cup [\bar{t} + \varepsilon, t_f] \\ \hat{u}, & t \in [\bar{t}, \bar{t} + \varepsilon) \end{cases}$$

where ε is any sufficient small positive real number, and $\hat{u} \in U_r$ is any r -dimension constant vector,

$$\text{setting } \Delta_{\varepsilon t} u(t) = \begin{cases} 0, & t \in [t_0, \bar{t}) \cup [\bar{t} + \varepsilon, t_f] \\ \hat{u} - u^*(t), & t \in [\bar{t}, \bar{t} + \varepsilon) \end{cases}$$

If $\Delta_{\varepsilon t} x(t), \Delta_{\varepsilon t} J[u^*(\cdot)]$ are the change quantity of $x^*, J[u^*(\cdot)]$ corresponding to $\Delta_{\varepsilon t} u(t)$, then from (6), $\Delta_{\varepsilon t} x(t)$ will result

$$\Delta_{\varepsilon t} x(t) = 0, \quad t \in [t_0, \bar{t}], \quad (11)$$

$$\begin{cases} \frac{d}{dt} \Delta_{\varepsilon t} x(t) = f(x^*(t) + \Delta_{\varepsilon t} x(t), \hat{u}(t)) - f(x^*(t), u^*(t)), & t \in [\bar{t}, t_f], \\ \Delta_{\varepsilon t} x(t) = 0, & t \in [\bar{t}, \bar{t} + \varepsilon), \\ \Delta_{\varepsilon t} x(t + \varepsilon) = \text{initiable value}, & t \in [\bar{t} + \varepsilon, t_f] \end{cases}, \quad (12)$$

Therefore

$$\begin{aligned} \Delta_{\varepsilon t} J[u^*(\cdot)] &= - \int_{\bar{t}}^{\bar{t} + \varepsilon} \frac{\partial F(x^*(t), \hat{u}(t), u^*(t), \psi(t))}{\partial x} \Delta_{\varepsilon t} x(t) dt \\ &\quad - \int_{\bar{t}}^{\bar{t} + \varepsilon} F(x^*(t), \hat{u}(t), u^*(t), \psi(t)) dt + \int_{t_0}^{t_f} o(\|\Delta_{\varepsilon t} x(t)\|) dt + o(\|\Delta_{\varepsilon t} x(t_f)\|). \end{aligned} \quad (13)$$

As $t \in [\bar{t}, \bar{t} + \varepsilon]$, note (4) and Lemma.1, and from (13) we can obtain directly that

$$\frac{d}{dt} \|\Delta_{\varepsilon t} x(t)\| \leq \left\| \frac{d}{dt} \Delta_{\varepsilon t} x(t) \right\| \leq a + b \|\Delta_{\varepsilon t} x(t)\|$$

Since $\Delta_{\varepsilon t} x(t) = 0$, then integration both sides above formula from \bar{t} to t

$$\|\Delta_{\varepsilon t} x(t)\| \leq a(t - t_f) + b \int_{t_f}^t \|\Delta_{\varepsilon t} x(\sigma)\| d\sigma \leq a\varepsilon + \int_{t_f}^t \|\Delta_{\varepsilon t} x(\sigma)\| d\sigma$$

and with the Lemma.2., we get

$$\|\Delta_{\varepsilon t} x(t)\| \leq a\varepsilon e^{b(t-t_f)} \leq a\varepsilon e^{b\varepsilon} = O(\varepsilon), \quad t \in [\bar{t}, \bar{t} + \varepsilon]. \quad (14)$$

And (14) implies that $\|\Delta_{\varepsilon t} x(t)\|$ is at most the same order as ε on the interval

$[\bar{t}, \bar{t} + \varepsilon]$. As $t \in [\bar{t}, \bar{t} + \varepsilon]$, through Lemma1. and (14), we can directly get

$$\frac{d}{dt} \|\Delta_{\varepsilon t} x(t)\| \leq \left\| \frac{d}{dt} \Delta_{\varepsilon t} x(t) \right\| \leq b \|\Delta_{\varepsilon t} x(t)\|,$$

then integrate both sides of the inequality from $\bar{t} + \varepsilon$ to t , and get

$$\|\Delta_{\varepsilon t} x(t)\| \leq \|\Delta_{\varepsilon t} x(\bar{t} + \varepsilon)\| + b \int_{\bar{t}}^t \|\Delta_{\varepsilon t} x(\sigma)\| d\sigma. \text{ According to Lemma.2., we get}$$

$$\|\Delta_{\varepsilon t} x(t)\| \leq \|\Delta_{\varepsilon t} x(\bar{t} + \varepsilon)\| e^{b[t - (\bar{t} + \varepsilon)]} \leq \|\Delta_{\varepsilon t} x(\bar{t} + \varepsilon)\| e^{b[t_f - (\bar{t} + \varepsilon)]} \leq O(\varepsilon), \forall t \in [\bar{t} + \varepsilon, t_f]. \quad (15)$$

Combination with (11), (12), (14) and (15), we gain $\|\Delta_{\varepsilon t} x(t)\| \leq O(\varepsilon), \forall t \in [t_0, t_f]$,

and submit it to (14), then $\Delta_{\varepsilon t} J[u^*(\cdot)] = - \int_{\bar{t}}^{\bar{t} + \varepsilon} F(x^*(t), \hat{u}(t), u^*(t), \psi(t)) dt + o(\varepsilon)$

Since $F(\cdot, \cdot, \cdot, \cdot)$ is continuous and $u^*(t)$ is continuous at \bar{t} , from the above we can get

$$\Delta_{\varepsilon t} J[u^*(\cdot)] = -\varepsilon F(x^*(\bar{t}), \hat{u}(\bar{t}), u^*(\bar{t}), \psi(\bar{t})) + o(\varepsilon) = -\varepsilon [H(x^*(\bar{t}), \hat{u}, \psi(\bar{t})) - H(x^*(\bar{t}), u^*(\bar{t}), \psi(\bar{t}))] + o(\varepsilon).$$

And $u^*(t)$ is the optimal control, therefore, for any $\Delta_{\varepsilon t} u(t)$ as the form of (10),

$$\Delta_{\varepsilon t} J[u^*(\cdot)] \geq 0, \text{ namely, } -\varepsilon [H(x^*(\bar{t}), \hat{u}, \psi(\bar{t})) - H(x^*(\bar{t}), u^*(\bar{t}), \psi(\bar{t}))] + o(\varepsilon) \geq 0,$$

and $\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0$. Hence $H(x^*(t), \hat{u}, \psi(t)) \leq H(x^*(t), u^*(t), \psi(t))$. Since $\hat{u} \in U_r$ is any r -dimension vector and $u^*(t)$ is continuous on the interval $[t_0, t_f]$, therefore,

$$H(x^*(t), u^*(t), \psi(t)) = \max_{u \in U_r} H(x^*(t), u(t), \psi(t)), \text{ and the proposition holds.}$$

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