

Analysis of Dynamic Characteristics of a New Lorenz-like Attractor

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Abstract. In this paper, a new three dimensional Lorenz-like chaotic system is reported. Nonlinear characteristic and basic dynamic properties of the three-dimensional autonomous system are studied by means of nonlinear dynamics theory, including the stability and the conditions for generating Hopf bifurcation of the equilibrium point. The chaotic system is demonstrated by numerical simulation.

Introduction

Nonlinear dynamics, commonly called the chaos theory, changes the scientific way of looking at the dynamics of natural and social systems, which has been intensively studied during the last five decades.

Chaos is a very interesting phenomenon closely related to nonlinear systems, which occurs so frequently that it has become important for workers in many disciplines to have a good grasp of the fundamentals and basic tools of the science of chaotic dynamics.

All this work stems ultimately from the original investigation of an extraordinary three-dimensional nonlinear system by the mathematical meteorologist Lorenz. In 1963, he discovered chaos in a simple system of three autonomous ordinary differential equations in order to describe the simplified Rayleigh-Benard problem [1]. It is notable that the Lorenz system has seven terms on the right-hand side, two of which are nonlinear (xz , xy). In 1999, Chen found a similar but nonequivalent chaotic attractor[2], which is now known to be the dual of the Lorenz system, in a sense defined[3]: when expressing the system in linear and nonlinear where the linear part has a constant matrix $A=[a_{ij}]_{3 \times 3}$, the Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while the Chen system satisfies $a_{12}a_{21} < 0$, Lü and Chen reported a new chaotic system called Lü system[4], which satisfies the condition $a_{12}a_{21} = 0$ and bridges the gap between the Lorenz and Chen systems. In 2002, a unified chaotic system was found that connects the newly found Chen's chaotic system to the classical Lorenz chaotic system through the Lü chaotic system [5].

In this paper, a new Lorenz-like chaotic system derived from the Lorenz system is proposed. It's a three-dimensional autonomous system which has six terms on the right-hand side of the governing equations, but only relies on two multipliers to introduce the nonlinearity necessary for folding trajectories.

Dynamical Behaviors of the New Chaotic Attractor

The new simpler chaotic system derived from the Lorenz system is given by:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = abx - axz \\ \dot{z} = xy - cz \end{cases} \quad (1)$$

Where $x = (x, y, z)^T \in R^3$ is the state variables of the system, a, b, c are constants and $a \neq 0$. There are six terms on the right-hand side but only relies on two quadratic nonlinearities xz and xy .

Some Basic Properties

Symmetry and Invariance. First, note that the system has a symmetry S because the transformation: $S : (x, y, z) \rightarrow (-x, -y, -z)$, which permits system invariant for all values of the system parameters a, b, c . Obviously, the z -axis itself is an orbit, that is, if $x=0, y=0$ at $t=0$, then $x=0, y=0$ for all $t > 0$. Moreover, the orbit on the z -axis tends to the origin when $t \rightarrow 0$. And the transformation S indicates that the system is symmetrical on the z -axis. Namely, both φ and S^φ are the solution of the system.

Dissipativity and the Existence of Attractor. It is easy to verify that the system is globally, uniformly, and asymptotically stable about its zero equilibrium point if parameters a, b, c satisfy $a > 0, b < 0, c > 0$. The Lyapunov function is constructed as:

$$V(x, y, z) = -bx^2 + y^2 + az^2 \quad (2)$$

which gives:

$$\begin{aligned} V(x, y, z) &= -2bx\dot{x} + 2y\dot{y} + 2az\dot{z} = -2bx(a(y-x)) + 2y(ax(b-z)) \\ &+ 2az(xy-cz) = 2a(bx^2 - cz^2) < 0 \end{aligned} \quad (3)$$

At the same time, the system can be a dissipative system, because the divergence of the vector field (4), also called the trace (6) of the Jacobian matrix (5) is negative if and only if the sum of the parameters a and c is positive, that is $a + c > 0$:

$$\nabla V = \frac{1}{V} \frac{d\vec{V}}{dt} = \text{div } \vec{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} \quad (4)$$

$$J = \begin{bmatrix} -a & a & 0 \\ a(b-z) & 0 & -ax \\ y & x & -c \end{bmatrix} \quad (5)$$

$$\text{Tr}(J) = -(a+c) \quad (6)$$

Furthermore, as for a chaotic system with n variables, there will be n Lyapunov exponents. And the sum of all these Lyapunov exponents is the average rate at which a cluster of initial conditions expands in n -dimensional hypervolume [6]. So the system has three Lyapunov exponents, the relation between the sum of three Lyapunov exponents and the divergence of the vector field is:

$$\nabla V = \frac{1}{V} \frac{d\vec{V}}{dt} = \text{div } \vec{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = \text{Tr}(J) = -(a+c) = \sum_{i=1}^3 \lambda_i = \sum LEs \quad (7)$$

$$V(t) = V(0)e^{-(a+c)t} \quad (8)$$

Where, $\lambda_i (i=1,2,3)$ are latent roots of the Jacobian matrix (5), LEs denote the three Lyapunov exponents of the system. So, the system will always be dissipative if and only if when $a + c > 0$ with an exponential rate:

$$\frac{d\vec{V}}{dt} = e^{-(a+c)t} \quad (9)$$

This is, a volume element $V(0)$ is contracted by the flow into a volume element $V(0)e^{-(a+c)t}$ in time t . This means that each volume containing the system trajectory shrinks to zero as $t \rightarrow \infty$ at an exponential rate $-(a+c)$. Therefore, all system orbits are ultimately confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor. This suggests that the dynamics may tend to an attractor when $t \rightarrow \infty$.

The Stability and the Bifurcation of the Equilibrium Points

The system has three equilibrium points: $O(0,0,0)$, $P_+ = (\sqrt{bc}, \sqrt{bc}, b)$, $P_- = (-\sqrt{bc}, -\sqrt{bc}, b)$ if $bc > 0$; but if $bc < 0$, there is only one fixed point $O(0,0,0)$; As for $bc = 0$, there is only one fixed point but which dependent on the value of the parameter b , which is denoted as $(0,0,b)$, $b \in \mathbb{R}$.

In this paper, the stability of the three equilibriums O, P_+, P_- are discussed only for $bc > 0$. Pitch fork bifurcation of the null solution at $b=0$ can be observed, if a, c are fixed while varying b . The other equilibriums P_+, P_- are symmetrically placed with respect to the z -axis.

Proposition 1: The fixed point O is always unstable for $a \neq 0, b > 0$.

Proof: At the fixed point $O(0,0,0)$, system (1) is linearized, the Jacobian matrix is defined as:

$$J_0 = \begin{bmatrix} -a & a & 0 \\ ab & 0 & -ax \\ y & x & -c \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} -a & a & 0 \\ ab & 0 & 0 \\ 0 & 0 & -c \end{bmatrix} \quad (10)$$

The characteristic polynomial is

$$\lambda^3 + (a+c)\lambda^2 + (ac - a^2b)\lambda - a^2bc = 0 \quad (11)$$

and the three eigenvalues corresponding to the fixed point O are:

$$\lambda_1 = -c, \quad \lambda_{2,3} = -\frac{a}{2} \pm \sqrt{ba^2 + \frac{a^2}{4}}.$$

So, it's clear that so far as the parameter $a > 0$, $\lambda_2 = \frac{a}{2}(-1 + \sqrt{4b+1}) > 0$; while $a < 0$,

$\lambda_3 = \frac{a}{2}(-1 - \sqrt{4b+1}) > 0$ both with $b > 0$. Consequently the fixed point $O(0,0,0)$ is unstable for $a \neq 0, b > 0$.

In the following, we consider the stability of the system at the fixed points P_+, P_- . Because the system is invariant under the transformation S , so one only needs to consider the stability of any one of the both. The stability of the system at the fixed points P_+ is analyzed in this paper.

Under the linear transformation $T: (x, y, z) \rightarrow (X, Y, Z)$

$$T: \begin{cases} x = X + \sqrt{bc} \\ y = Y + \sqrt{bc} \\ z = Z + b \end{cases} \quad (12)$$

The system (1) becomes

$$\begin{cases} \dot{X} = a(Y - X) \\ \dot{Y} = -a(X + \sqrt{bc})Z \\ \dot{Z} = (X + \sqrt{bc})(Y + \sqrt{bc}) - c(Z + b) \end{cases} \quad (13)$$

The fixed point P_+ of the system (1) is switched to the new equilibrium $O'(0,0,0)$ of the system (13) under the linear transformation. In the following, the stability of the system (13) at the fixed points O' is considered.

The Jacobian matrix of the system (13) at $O'(0,0,0)$ is (14), with the characteristic polynomial is (15):

$$J_0 = \begin{bmatrix} -a & a & 0 \\ -az & 0 & -a\sqrt{bc} \\ Y + \sqrt{bc} & X + \sqrt{bc} & -c \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & -a\sqrt{bc} \\ \sqrt{bc} & \sqrt{bc} & -c \end{bmatrix} \quad (14)$$

$$\lambda^3 + (a+c)\lambda^2 + (ac + abc)\lambda + 2a^2bc = 0 \quad (15)$$

Using Routh-Hurwitz criterion, the equation (15) has all roots with negative real parts if and only if the conditions are satisfied as follows:

$$\begin{cases} a + c > 0 \\ (a + c)(ac + abc) - 2a^2bc > 0 \end{cases} \quad (16)$$

So, the fixed points P_+ and P_- of the system (1) are asymptotically stable if and only if any one of the conditions (16) is satisfied. Furthermore, it's easy to verify that the equation (15) has a pair of conjugate purely imaginary eigenvalues and one real negative eigenvalue if and only if the conditions are satisfied as follows:

$$\begin{cases} a + c > 0 \\ ac(b+1) > 0 \\ a + c + bc = ab \end{cases} \quad (17)$$

And the real root $\lambda_1 = -(a + c)$, the two imaginary roots $\lambda_{2,3} = \pm i\sqrt{ac(b+1)}$.

Consequently, we have the following conclusion:

Proposition 2: Equation (15) has a negative real root $\lambda_1 = -(a + c)$ together with a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm i\sqrt{\frac{2a^2c}{a-c}}$, and $\text{Re}(\lambda'_c(c)) \neq 0$, therefore, the system (1) displays a Hopf bifurcation at the point P_+ if conditions (17) are satisfied [7] [8].

Proof: let $\Theta = (X, Y, Z)^T$, then we get

$$\Theta = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} a(Y - X) \\ -a(X + \sqrt{bc})Z \\ (X + \sqrt{bc})(Y + \sqrt{bc}) - c(Z + b) \end{bmatrix} = f(\Theta, a, b, c) \quad (18)$$

It's very easy to verify that $f(0, a, b, c) = 0$ for $\forall a, b, c \in R$ and $b \neq \pm 1$ from (17). The interrelation of the parameters a , b and c is:

$$a = \frac{c(b+1)}{b-1}, \quad b = \frac{a+c}{a-c}, \quad c = \frac{a(b-1)}{b+1} \quad (19)$$

Therefore, from the equation characteristic polynomial (15), we have:

$$\lambda'_c = -\frac{\lambda^2 + a(b+1)\lambda + 2a^2b}{3\lambda^2 + 2(a+c)\lambda + ac(b+1)} \quad (20)$$

$$\lambda'_c(c) = -\frac{\lambda^2 + a(b+1)\lambda + 2a^2b}{3\lambda^2 + 2(a+c)\lambda + ac(b+1)} \Big|_{c=\frac{a(b-1)}{b+1}} = -\frac{(b+1)\lambda^2 + a(b+1)^2\lambda + 2a^2b(b+1)}{3(b+1)\lambda^2 + 4ab\lambda + a^2(b^2-1)} \quad (21)$$

Substituting $\lambda_{2,3} = \pm i\sqrt{ac(b+1)}$, the real part and imaginary part of the $\lambda'_c(c)$ respectively are:

$$\text{Re}(\lambda'_c(c)) = -\frac{b^3 + b^2 - b - 1}{3b^2 + 10b^2 - 2b - 2} = -\frac{(b^2 - 1)(b + 1)}{2b^3 + 10b^2 - 2b - 2} \neq 0$$

$$\text{Im}(\lambda'_c(c)) = -\frac{\sqrt{b-1}(b^4 - 2b^2 + 1)}{2b^4 + 8b^3 - 12b^2 + 2} \neq 0$$

Consequently, the system (13) displays a Hopf bifurcation at $O'(0, 0, 0)$, so the system (1) displays a Hopf bifurcation at $P_+ = (\sqrt{bc}, \sqrt{bc}, b)$.

However, it is easy to actually verify that there does not exist a diffeomorphism between the system (1) and the other Lorenz-like systems, since their eigenvalues of the corresponding Jacobians are not equivalent, and furthermore, they are not topological equivalent. Since the verification is straight forward but rather tedious, it is omitted here. Similarly, it is easy to verify that the system is not topological equivalent to the unified system and other Lorenz-shaped systems.

Simulation and Analysis

For the system (13), set $a=5$, $b=4$, $c=2$, the initial condition $(-11.2, -8.4, 33.4)$, simulation time 50 s, the system (13) is dissipative and exits a chaotic attractor for the parameter values. Fig.1 shows the phase space orbit in x - y plane, and Fig. 2 shows the chaotic attractor in the x - z plane.

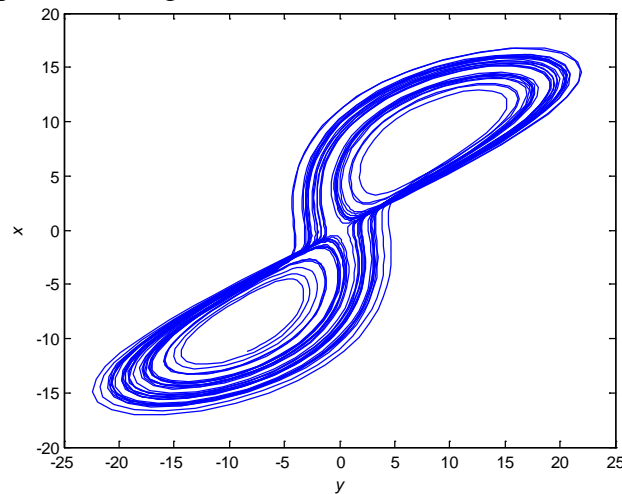


Fig.1 Phase diagram in x - y plane of system (13)

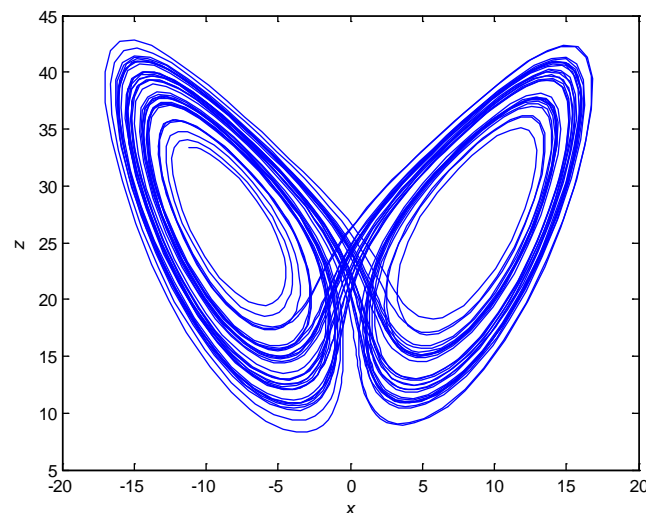


Fig.2 Phase diagram in x - z plane of system (13)

Further analyze, the three Lyapunov exponents LEs is $(0.6262, 0, -7.6262)$. So the sum of LEs is $\sum LEs = -7 = -(a+c) = Tr(J)$, and the Lyapunov dimension $D=2.0821$. At the same time, all the three equilibrium points O, P_+, P_- are unstable.

Take Fig.2 for example, we can see that the solution spirals outward from one of the equilibrium point P_+ or P_- for some times, then switches to spiraling outward from the other equilibrium point. This pattern repeats forever with the number of circuits around an equilibrium point before switching appearing to vary erratic manner.

Conclusions

In this paper, a new chaotic system is proposed and studied in detail by varying three control parameters. Dynamical behaviors of the new system are analyzed through both theoretically and numerically.

There are abundant and complex dynamical behaviors which the new autonomous system produces, despite its apparent simplicity is investigated and expatiated in this paper, but the attractors and their forming mechanism need study and explore further, and their topological structure should

be completely and thoroughly investigated. Therefore, further research into the system is still important and insightful.

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