Abstract—In this paper, we introduce a split general mixed variational inequality problem, which is a natural extension of a split variational inequality problem, split general quasi-variational inequality problem in Hilbert spaces. Using the resolvent operator technique, we propose two classes of perturbed iterative algorithms for the split general mixed variational inequality problem. Further, we discuss the convergence criteria of the iterative algorithms. The results presented here extend and improve many previously known results in this area.

Key word—split general mixed variational inequality problem; split general quasi-variational inequality problem; perturbed iterative algorithm; convergence

I. INTRODUCTION

In a recent paper [1], Kazmi has developed an iterative algorithm for finding approximate solution for a new split general quasi-variational inequality problem in Hilbert spaces. The aim of this work is to extend his idea to more general problem. Throughout the paper unless stated otherwise, for each \( i \in \{1,2\} \), let \( H_i \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( f_i : H_i \rightarrow H_i, g_i : H_i \rightarrow H_i \) be continuous mappings with \( \text{Im} \, g_i \cap \text{dom} \, \phi_i \neq \emptyset \). Let \( A : H_1 \rightarrow H_2 \) be a bounded linear operator with its adjoint operator \( A^* \). We consider the following problem: Find \( x^*_i \in H_i \) such that \( g_i(x^*_i) \in \text{dom} \, \phi_i \) and

\[
\langle f_i(x^*_i), x_i - g_i(x^*_i) \rangle \geq \phi_i(g_i(x^*_i)) - \phi_i(x_i), \quad \forall x_i \in H_i,
\]

(1)

and such that \( x_2^* = A x_1^* \in H_2 \), \( g_2(x_2^*) \in \text{dom} \, \phi_2 \) solves

\[
\langle f_2(x_2^*), x_2 - g_2(x_2^*) \rangle \geq \phi_2(g_2(x_2^*)) - \phi_2(x_2), \quad x_2 \in H_2.
\]

(2)

We call problem (1)-(2) the split general mixed variational inequality problem (in short, SpGMVIP). SpGMVIP (1)-(2) amounts to saying: find a solution of general mixed variational inequality problem (1) image under a given bounded linear operator is a solution of general mixed variational inequality problem (2). For convenience, we denote the solution set of SpGMVIP(1)-(2) by

\[
\Gamma = \{ x^*_i \in H_i \mid x^*_i \text{ solves } (1) \text{ and } A x^*_1 \in H_2 \text{ solves } (2) \}.
\]

Next, we give some special cases of SpGMVIP (1)-(2).

If we set \( g_i = I_i \), where \( I_i \) is an identity operator on \( H_i \), then SpGMVIP (1)-(2) is reduced to the following split mixed variational inequality problem (in short, SpMVIP): Find \( x^*_i \in H_i \) such that

\[
\langle f_i(x^*_i), x_i - x^*_i \rangle \geq \phi_i(x^*_i) - \phi_i(x_i), \forall x_i \in H_i,
\]

(3)

and such that \( x^*_2 = A x^*_1 \in H_2 \) solves

\[
\langle f_2(x^*_2), x_2 - x^*_2 \rangle \geq \phi_2(x^*_2) - \phi_2(x_2), \forall x_2 \in H_2,
\]

(4)

which appears to be new one.

If we set \( \phi_i(0) = \delta_{C_i}(x^*_i) = \delta_{C_i \cap \text{dom} \, \phi_i}(x^*_i) \), \( m : H_i \rightarrow H_i \) is a single-valued mapping, where \( C_i(x^*_i) = C_i + m(x^*_i) \), and \( C_i \) is a closed convex subset of \( H^*_i \); then SpMVIP (1)-(2) is reduced to the following split general quasi-variational inequality problem (in short, SpGQVIP): Find \( x^*_i \in H_i \), such that \( g_i(x^*_i) \in C_i(x^*_i) \) and

\[
\langle f_i(x^*_i), x_i - g_i(x^*_i) \rangle \geq 0, \forall x_i \in C_i(x^*_i),
\]

(5)

and such that \( x^*_2 = A x^*_1 \), \( g_2(x^*_2) \in C_2(x^*_2) \) solves

\[
\langle f_2(x^*_2), x_2 - g_2(x^*_2) \rangle \geq 0, \forall x_2 \in C_2(x^*_2).
\]

(6)

This problem was introduced and studied by Kazmi in [1] and he exhibited split quasi-variational inequality problem, split general variational inequality problem and quasi-variational inequality problem as special cases of SpGQVIP (5)-(6). For details, see reference [1].

If \( \phi_i = \delta_{C_i} \), the indicator function of a closed convex set \( C_i \subset H_i; g_i = I_i \), the identity mapping \( H_i \), then SpMVIP (1)-(2) is reduced to the following split general variational inequality problem (in short, SPVIP): Find \( x^*_i \in H_i \), such that

\[
\langle f_i(x^*_i), x_i - x^*_i \rangle \geq 0, \forall x_i \in C_i,
\]

(7)

and \( x^*_2 = A x^*_1 \in H_2 \) solves

\[
\langle f_2(x^*_2), x_2 - x^*_2 \rangle \geq 0, \forall x_2 \in C_2,
\]

(8)
which has been introduced and studied by Censor, Gibali and Reich[2]. It is worth noting that SpGMVIP(1)-(2) is quite general and includes as special cases split minimization between two spaces so that the image of a minimizer of a given function, under a bounded linear operator, is a minimizer of a given function, under a bounded linear operator, is a minimizer of another function, split zero problem and the split feasibility problem which have already been studied and used in practice as a model in the intensity-modulated radiation therapy planning, see[3, 4, 5].

In a word, SpGMVIP is more general, which is one of our motivations to write this paper. By using the resolvent operator technique about the maximal monotone mapping, we propose two classes of perturbed iterative algorithms taking into account a possible inexact computation for SpGMVIP (1)-(2) and discuss the convergence criteria of these iterative algorithms. The results presented here extend and improve the previously known results in this area.

II. PERTURBED ITERATIVE ALGORITHMS

To begin with, let us transform SpGMVIP (1)-(2) into fixed point problems.

**Lemma 2.1.** \(x^*_i \in \Gamma\) if and only if \(x^*_i\) satisfies the following relations

\[
g_i(x^*_i) = J^\infty_{\rho^*_i}(g_i(x^*_i) - \rho^*_i f_i(x^*_i)),
\]

\[
g_i(Ax^*_i) = J^\infty_{\rho^*_i}(g_i(x^*_i) - \rho^*_i f_i(Ax^*_i)),
\]

where \(\rho^*_i > 0\) is a constant and \(J^\infty : H \rightarrow R \cup \{\pm \infty\}\) is the resolvent operator of the maximal monotone mapping \(\partial \phi\).

Noting that \(\partial \phi\) denotes the subdifferential of a proper, convex and lower semi-continuous function \(\phi : H \rightarrow R \cup \{\pm \infty\}\),

**Proof.** From definition of \(J^\infty\), it follows from (9) that

\[
g_i(x^*_i) - \rho^*_i f_i(x^*_i) \in g_i(x^*_i) + \rho^*_i \partial \phi_i(g_i(x^*_i))
\]

then \(-f_i(x^*_i) \in \partial \phi_i(g_i(x^*_i))\), definition of \(\partial \phi\) implies

\[
\phi_i(x_i) \geq \phi_i(g_i(x^*_i)) + \langle -f_i(x^*_i), x_i - g_i(x^*_i) \rangle, \forall x_i \in H_i,
\]

this is,

\[
\langle f_i(x^*_i), x_i - g_i(x^*_i) \rangle \geq \phi_i(g_i(x^*_i)) - \phi_i(x_i), \forall x_i \in H_i
\]

thus \(x^*_i\) is a solution of (1). Similarly, it is easy to know \(Ax^*_i\) solves (2), hence \(x^*_i \in \Gamma\). The converse relation is obvious, so is omitted, completing the proof.

Based on Lemma 2.1, we can propose the following perturbed iterative algorithms for approximating a solution to SpGMVIP(1)-(2). Let \(\{\alpha^*_n\} \subseteq (0,1)\) be a sequence such that \(\sum_{\alpha^*_n = 0}^\infty = \infty\) and let \(\rho_1, \rho_2, \gamma\) be the parameters with positive values.

**Algorithm 2.1.** Given \(x^0_i \in H_i\), compute the iterative sequence \(\{x^*_i\}\) defined by the iterative schemes

\[
g_i(y^*) = J^\infty_{\rho^*_i}(g_i(x^*_i) - \rho^*_i f_i(x^*_i)),
\]

\[
g_i(z^*) = J^\infty_{\rho^*_i}(g_i(Ay^*) - \rho^*_i f_i(Ay^*)),
\]

\[
x^*_i + 1 = (1 - \alpha^*_i)x^*_i + \alpha^*_i [y^* + \gamma A^*(z^* - Ay^*)] + \alpha^*_i e_i^*,
\]

for all \(n = 0, 1, 2, \cdots, \rho_1, \rho_2, \gamma > 0\) and take into account a possible inexact computation, an \(\|\| \rightarrow 0(n \rightarrow \infty)\) error \(e_i^*\) is added in the right hand side of (13) with Moreover, we consider other perturbations by replacing in (11) and (12) \(\phi_i\) by \(\phi_i^\gamma\), where the sequence \(\{\phi_i^\gamma\}\) approximates \(\phi_i\), \(\{\phi_i^\gamma\}\) is a collection of proper convex semi-continuous functions on \(H_i\).

If \(\phi_i^\gamma = \delta_{C_i^\gamma}^\gamma\), where \(C_i^\gamma(x^*_i)\) is same as the above, \(e_i^* = 0\), then Algorithm 2.1 is reduced to the following algorithm for SpGQVIP:

**Algorithm 2.2.** Given \(x^0_i \in H_i\), compute the iterative sequence \(\{x^*_i\}\) defined by the iterative schemes

\[
g_i(y^*) = P_{C_i^\gamma}(g_i(x^*_i) - \rho^*_i f_i(x^*_i)),
\]

\[
g_i(z^*) = P_{C_i^\gamma}(g_i(Ay^*) - \rho^*_i f_i(Ay^*)),
\]

\[
x^*_i + 1 = (1 - \alpha^*_i)x^*_i + \alpha^*_i [y^* + \gamma A^*(z^* - Ay^*)],
\]

for all \(n = 0, 1, 2, \cdots, \rho_1, \rho_2, \gamma > 0\), where \(P_{C_i^\gamma}\) is the metric projection of \(H_i\) on to \(C_i^\gamma\), and it is well known that \(P_{C_i^\gamma}\) is a nonexpansive mapping. Algorithm 2.2 was proposed by Kazmi [1] for SpGQVIP.

Observe that (9) and (10) can change into the following:

\[
x^*_i = x^*_i - g_i(x^*_i) + J^\infty_{\rho^*_i}(g_i(x^*_i) - \rho^*_i f_i(x^*_i)), i = 1, 2,
\]

where \(x^*_i = Ax^*_i\), \(\rho^*_i > 0\) is a constant. In view of the above equations, we can propose another perturbed iterative algorithm for SpGMVIP.

**Algorithm 2.3.** Given \(x^0_i \in H_i\), compute the iterative sequence \(\{x^*_i\}\) defined by the iterative schemes

\[
y^* = x^*_i - g_i(x^*_i) + J^\infty_{\rho^*_i}(g_i(x^*_i) - \rho^*_i f_i(x^*_i)),
\]

\[
z^* = Ay^* - g_i(y^*) + J^\infty_{\rho^*_i}(g_i(Ay^*) - \rho^*_i f_i(Ay^*)},
\]
\[
\alpha_{n+1} = (1 - \alpha^n) \alpha_n + \alpha^n [y^n + \gamma \mathcal{A}^* (z^n - A y^n)] + \alpha^n e^n, \tag{19}
\]
for all \( n = 0, 1, 2, \ldots, \rho, \rho, \rho, \rho > 0 \), \( e^n \) is an error term and \( \|e^n\| \to 0 (n \to \infty) \).

In order to obtain our main results, we need the following definition, Assumption and lemmas.

Definition 2.1. A nonlinear mapping \( f: H_1 \to H_1 \) is said to be

(i) \( \alpha \)-strongly monotone if there exists a constant \( \alpha > 0 \) such that

\[
\langle f(x) - f(y), x - y \rangle \geq \alpha \|x - y\|^2, \forall x, y \in H_1.
\]

(ii) \( \beta \)-Lipschitz continuous if there exists a constant \( \beta > 0 \) such that

\[
\|f(x) - f(y)\| \leq \beta \|x - y\|, \forall x, y \in H_1.
\]

Remark 2.1. It is easy to know that if \( f: H_1 \to H_1 \) is \( \alpha \)-strongly monotone and \( \beta \)-Lipschitz continuous then \( \alpha \leq \beta \).

Assumption 2.2. For \( i \in [1, 2] \), let \( \varphi_i: H_i \to \mathbb{R} \cup \{+ \infty\} \) be a proper, convex and lower semi-continuous function, \( \{\varphi_i^*\} \) approximate \( \varphi_i \) and satisfies the condition:

\[
\lim_{k \to +\infty} \int_{\varphi_i^*(v_i) - J_{\varphi_i^*}^\infty (v_i)} = 0, \forall v_i \in H_i.
\]

Lemma 2.3([6]). Let \( \{a_k\} \) be a sequence of nonnegative real numbers satisfying the condition

\[
a_{k+1} \leq (1 - m_k) a_k + m_k \delta_k, \forall k \geq 0.
\]

where \( \{m_k\}, \{\delta_k\} \) are sequences of real numbers such that

(i) \( \{m_k\} \subset [0, 1] \) and \( \sum_{k=0}^{\infty} m_k = \infty \), or, equivalently,

\[
\prod_{k=0}^{\infty} (1 - m_k) = \lim_{k \to +\infty} \prod_{j=0}^{k} (1 - m_j) = 0;
\]

(ii) \( \limsup_{k \to +\infty} \delta_k \leq 0 \), or (ii) \( \sum_{k=0}^{\infty} \delta_k m_k \) is convergent.

Then \( \lim_{k \to +\infty} a_k = 0 \).

Lemma 2.4. Let \( H \) be a real Hilbert space, for all \( x, y \in H \), the following hold:

\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle x, y \rangle,
\]

\[
\|x + y\|^2 = \|x\|^2 + 2 \langle x, y \rangle + \|y\|^2.
\]

III. MAIN RESULTS

Theorem 3.1. For each \( i \in [1, 2] \), let \( g_i: H_i \to H_i \) be \( \delta_i \)-Lipschitz continuous such that \( (g_i - I_i) \) is \( \delta_i \)-strongly monotone, where \( I_i \) is the identity operator on \( H_i \). Let \( f_i: H_i \to H_i \) be \( \alpha_i \)-strongly monotone with respect to \( g_i \) and \( \beta_i \)-Lipschitz continuous. Let \( \mathcal{A}: H_1 \to H_2 \) be a bounded linear operator and let \( \mathcal{A}^* \) be its adjoint operator. Suppose \( x_i^* \in \Gamma \) and Assumption 2.2 holds. Then the sequence \( \{x_i\} \) generated by Algorithm 2.1 converges strongly to \( x_i^* \) provided that the constant \( \rho_i \) and \( \gamma \) satisfy the conditions

\[
\rho_i - \frac{\alpha_i}{\beta_i} < \sqrt{\alpha_i^i + \delta_i^i}, \delta_i < \sqrt{\alpha_i^i + \alpha_i^i}, \gamma \|d_i\| \|	heta_i\| \theta_i < 1 - \theta_i, \gamma \in \left(0, \frac{2}{\|d_i\|}\right),
\]

\[
\theta_i = \tau_i, \delta_i^i = 2 \rho_i \alpha_i + \beta_i^2, \tau_i = \frac{1}{\sqrt{1 + 2\sigma_i}}, \rho_i > 0, i = 1, 2.
\]

Proof. Since \( x_i^* \in \Gamma \), then \( x_i^* \in H_i \) such that

\[
g_i(x_i^*) \in \text{dom} \varphi_i(g_i(x_i^*)) \text{ and }
\]

\[
g_i(x_i^*) = J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*)),
\]

\[
g_i(Ax_i^*) = J_{\delta_i^i}^\infty (g_i(Ax_i^*) - \rho_i f_i(Ax_i^*)),
\]

for \( \rho_i > 0 \) and \( x_i^* = Ax_i^* \). From Algorithm 2.1(11), Assumption 2.2 and (20), we have

\[
\|g_i(x^n) - g_i(x_i^*)\| = \|J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*)) - J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*))\|
\]

\[
\leq \|J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*)) - J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*))\|
\]

\[
+ \|J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*)) - J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*))\|
\]

\[
\leq \|g_i(x_i^*) - g_i(x_i^*) - \rho_i (f_i(x_i^*) - f_i(x_i^*))\| + \|e_i^*\|,
\]

where \( e_i^* = \|J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*)) - J_{\delta_i^i}^\infty (g_i(x_i^*) - \rho_i f_i(x_i^*))\| \) and \( \lim_{k \to +\infty} e_i^* = 0 \) owns to Assumption 2.2. Now, using the facts that \( f_i \) is \( \alpha_i \)-strongly monotone with respect to \( g_i \) and \( \beta_i \)-Lipschitz continuous, and \( g_i \) is \( \delta_i \)-Lipschitz continuous, we have

\[
\|g_i(x^n) - g_i(x_i^*) - \rho_i (f_i(x^n) - f_i(x_i^*))\| = \|g_i(x^n) - g_i(x_i^*)\|
\]

\[
- 2 \rho_i \|f_i(x_i^*) - f_i(x_i^*)\| + \rho_i \|f_i(x_i^*) - f_i(x_i^*)\|
\]

\[
\leq (\delta_i^i - 2 \rho_i \alpha_i + \beta_i^2) \|e_i^* - x_i^*\|.
\]

Combining (22) and (23), we have
\[ \| (x^* - g_i(x_i^*)) \| \leq \sqrt{\delta_1^2 - 2\rho_1(\alpha_1 + \rho_1^2\beta_1^2)} \| x_i^* - x_i \| + \varepsilon_i. \]  

(24)

Since \((g_1 - I)\) is \(\sigma_1\)-strongly monotone, we have

\[ \| y^* - x_i^* \| \leq \| g_i(y^*) - g_i(x_i^*) \| - 2(g_1 - I)y^* - (g_1 - I)x_i^* \]
\[ \leq \| g_i(y^*) - g_i(x_i^*) \| - 2\sigma_1\| y^* - x_i^* \|. \]

which implies

\[ \| y^* - x_i^* \| \leq \tau_1\| g_i(y^*) - g_i(x_i^*) \|. \]  

(25)

where \( \tau_1 = \frac{1}{\sqrt{1 + 2\sigma_1}}. \) From (24) and (25), we get

\[ \| y^* - x_i^* \| \leq \theta_1\| y^* - x_i^* \| + \tau_1\varepsilon_i, \]  

(26)

where \( \theta_1 = \tau_1\sqrt{\delta_1^2 - 2\rho_1(\alpha_1 + \rho_1^2\beta_1^2)}. \) Similarly, from Algorithm 2.1(12), Assumption 2.2 and (21) and using the facts that \( f_2 \) is \( \alpha_2 \)-strongly monotone with respect to \( g_2 \)

and \( \beta_2 \)-Lipschitz continuous, \((g_2 - I)\) is \(\sigma_2\)-strongly monotone, and \( g_2 \) is \(\delta_2\)-Lipschitz continuous, we have

\[ \| g_2(z^*) - g_2(Ax_i^*) \| \leq \sqrt{\delta_2^2 - 2\rho_2(\alpha_2 + \rho_2^2\beta_2^2)}\| Ay^* - Ax_i^* \|. \]  

(27)

and

\[ \| y^* - Ax_i^* \| \leq \theta_2\| Ay^* - Ax_i^* \| + \tau_2\varepsilon_i, \]  

(28)

where

\[ \tau_2 = \frac{1}{\sqrt{2\delta_2^2 + 1}}, \quad \theta_2 = \tau_2\sqrt{\delta_2^2 - 2\rho_2(\alpha_2 + \rho_2^2\beta_2^2)}, \]

and \( \lim_{n \to \infty} \varepsilon_i = 0 \) owns to Assumption 2.2. From Algorithm 2.1(13), we obtain

\[ \| y^* - x_i^* \| \leq (1 - \alpha_i^*)\| y^* - x_i \| + \alpha_i^*\| y^* - x_i^* \| + \beta_i^*\| x_i^* - x_i \| + \gamma\| x_i^* - x_i \|. \]  

(29)

Further, using the definition of \( A^* \), the face that \( A^* \) is a bounded linear operator with \( \| A^* \| = \| A \| \), and the given condition on \( \gamma \), we have

\[ \| y^* - x_i^* \| \leq 2\gamma\| y^* - x_i^* \| + \beta_i^*\| x_i^* - x_i \| + \gamma\| x_i^* - x_i \| \]
\[ \leq \| y^* - x_i \| + \gamma(\gamma(\gamma - 1)\| A^* - A \| \leq \| y^* - x_i^* \|. \]  

(30)

and, using (3.9), we have

\[ \| y^* - Ax_i^* \| \leq \| A^* - A \| \leq \theta_3\| A^* - A \| \leq \| A^* - A \| + \| y^* - x_i^* \|. \]  

(31)

It follows from (29)-(31), we obtain

\[ \| x_{i+1}^* - x_i^* \| \leq (1 - \alpha_i^*)\| x_{i+1}^* - x_i \| + \alpha_i^*\| x_{i+1}^* - x_i \| + \gamma\| x_{i+1}^* - x_i \| \]
\[ + \beta_i^*\| x_i^* - x_i \| + \gamma\| x_i^* - x_i \| \]
\[ = (1 - \theta_i^*)\| x_{i+1}^* - x_i \| + \gamma\| x_{i+1}^* - x_i \| + \beta_i^*\| x_i^* - x_i \|. \]  

(32)

where \( \theta = \theta_i(1 + \gamma\| A \| \theta_i). \) It follows from the conditions on \( \rho_1, \rho_2 \)

and \( \gamma \) that \( \theta \in (0,1). \) Letting

\[ a_s = \| x_{i+1}^* - x_i^* \| - \alpha_s(1 - \theta), \]

\[ \delta_s = \frac{1}{1 - \theta_i}(1 + \gamma\| A \| \theta_i)\| x_{i+1}^* - x_i \| + \gamma\| x_{i+1}^* - x_i \| + \beta_i^*\| x_i^* - x_i \|, \]

where \( s = 1, 2 \), we have \( a_s = (1 - m_s)a_s + m_s\delta_s \). Moreover the conditions (i) and (ii) of Lemma 3.1 are satisfied. It follows that \( \{ x_i \} \) converges strongly to \( x_i^* \) as \( n \to \infty \).

Since \( A \) is continuous, it follows from (25), (26)-(28) that

\[ y^* \to x_i^*, \quad g_i(y^*) \to g_i(x_i^*), \quad Ay^* \to Ax_i^*, \quad z^* \to Ax_i^* \]

and \( g_2(z^*) \to g_2(Ax_i^*) \) as \( n \to \infty \). This completes the proof.

In the following, we consider the convergence of Algorithm 2.3 for SpGMVIP.

**Theorem 3.2.** For each \( i \in [0,2] \), let \( g_i : H_i \to H \) be \( \sigma_i \)-strongly monotone and \( \beta_i \)-Lipschitz continuous. Let \( f_i : H \to H_i \) be \( \alpha_i \)-strongly monotone with respect to \( g_i \) and \( \beta_i \)-Lipschitz continuous. Let \( A : H_i \to H \) be a bounded linear operator and \( A^* \) be its adjoint operator. Suppose \( x_i^* \in \Gamma \) and Assumption 2.2 holds. Then the sequence \( \{ x_i^* \} \) generated by Algorithm 2.3 converges strongly to \( x_i^* \) provided that the constants \( \rho \) and \( \gamma \) satisfy the conditions

\[ \rho > \beta_i\sqrt{k_i(2 - k_i)}, \quad k_i < 1, \]

\[ k_i = \sqrt{1 - 2\delta_i^2 + \delta_i^2}, \]

\[ \theta_i = \theta_i + \frac{1}{\sqrt{1 - 2\rho_i(\alpha_i + \rho_i^2\beta_i^2)}, \quad \rho_i > 0, \ i = 1,2, \]

\[ \rho_i = \theta_i \frac{\alpha_i}{\beta_i} + \frac{k_i}{k_i(2 - k_i)} \]

Proof Since \( x_i^* \in H_i \), then \( x_i^* \in H_i \) is such that \( g_i(x_i^*) \in dom \varphi_i \), and

\[ x_i^* = x_i^* - g_i(x_i^*) + J_{\rho_i}(g_i(x_i^*) - \rho_i^2f_i(x_i^*). \]  

(33)
\[A\hat{x}_i^t = A\hat{x}_i^{t-1} - g_i(A\hat{x}_i^{t-1}) + J_{\rho_i}^{\infty}(g_i(A\hat{x}_i^{t-1}) - \rho_i f_i(A\hat{x}_i^{t-1})) , \quad (34)\]

for \(\rho_i > 0\). From Algorithm 2.3(19), Assumption 2.2 and (33), we have

\[\|y' - x_i'\| \leq \|y' - x_i\| + \|\rho_i f_i(x_i') - f_i(x_i)\| \leq \|y' - x_i\| + \rho_i \|f_i(x_i') - f_i(x_i)\| \leq \|y' - x_i\| + \rho_i \|e_i'\| . \quad (35)\]

Note that \(g_i\) is \(\sigma_i\)-strongly monotone and \(\delta_i\)-Lipschitz continuous, and \(f_i\) is \(\sigma_i\)-strongly monotone with respect to \(g_i\) and \(\beta_i\)-Lipschitz continuous, we have

\[\|y' - x_i' - (g_i(x_i') - g_i(x_i))\| \leq k_i \|e_i'\| \quad (36)\]

where \(k_i = \sqrt{1 - 2\sigma_i + \delta_i^2}\),

\[g_i(x_i') - g_i(x_i) - \rho_i f_i(x_i') - f_i(x_i) \leq \sqrt{1 - 2\rho_i \sigma_i + \rho_i^2 \beta_i^2} \|e_i'\| \quad (37)\]

from (33)-(35), we have

\[\|y' - x_i'\| \leq \theta_i \|x_i' - x_i\| + \epsilon_i' , \quad (38)\]

where

\[\theta_i = k_i + \sqrt{1 - 2\rho_i \sigma_i + \rho_i^2 \beta_i^2}, \quad \epsilon_i' = \sqrt{\|J_{\rho_i}^{\infty}(g_i(x_i') - \rho_i f_i(x_i')) - J_{\rho_i}^{\infty}(g_i(x_i') - \rho_i f_i(x_i))\|} . \]

Similary , from Algorithm 2.3(18), Assumption 2.2 and (34), and using the fact that \(f_i\) is \(\sigma_i\)-strongly monotone with respect to \(g_i\) and \(\beta_i\)-Lipschitz continuous, \(g_i\) is \(\sigma_i\)-strongly monotone and \(\delta_i\)-Lipschitz continuous, we have

\[\|x' - Ax_i^t\| \leq \theta_i \|y' - Ax_i^t\| + \epsilon_i^t , \quad (39)\]

where \(\theta_i = k_\gamma + \sqrt{1 - 2\rho_\gamma \sigma_\gamma + \rho_\gamma^2 \beta_\gamma^2}, \quad k_\gamma = \sqrt{1 - 2\sigma_\gamma + \delta_\gamma^2}, \quad \epsilon_i^t = \sqrt{\|J_{\rho_i}^{\infty}(g_i(Ax_i^t) - \rho_i f_i(Ax_i^t)) - J_{\rho_i}^{\infty}(g_i(Ax_i^t) - \rho_i f_i(Ax_i^t))\|} . \]

Combining (29)-(31), (38), (39), we obtain

\[\|y_{i+1}^t - x_i^t\| \leq [1 - \alpha^* (1 - \theta)]\|y_i^t - x_i\| + \alpha^* (1 + \gamma \|\theta_i\|) \|e_i^t\| + \gamma \|e_i^t\| + \|e_i\| , \quad (40)\]

where \(\theta = \theta_i + \gamma \|\theta_i\| \theta_i \). It follows from the conditions on \(\rho_i\), \(\rho_\gamma\) and \(\gamma\) that \(\theta \in (0, 1)\). Letting

\[a_i = \|y_i^t - x_i^t\|, \quad m_i = \alpha^* (1 - \theta), \quad \delta_i = \frac{1}{1 - \theta} (1 + \gamma \|\theta_i\|) \|e_i^t\| + \gamma \|e_i^t\| + \|e_i\| , \forall n \geq 0 . \]

Then (40) implies

\[a_{i+1} \leq (1 - m_i) a_i + m_i \delta_i , \forall n \geq 0 . \]

Noting the conditions (i) and (ii) of Lemma 3.1 are satisfied and it follows that \(\{{x_i^t}\}\) converges strongly to \(x_i^*\) as \(n \to \infty\). The rest of argument is same as in Theorem 3.3, so is omitted, which is completed the proof.

Remark 3.3. Algorithm 2.2 is a special case of Algorithm 2.1 and Theorem 3.1 extends the corresponding results in [1].

REFERENCES


