Lacunary sequences that do not influence the uniqueness of solution of the inverse Borg–Levinson problem

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Abstract—Here we present the inverse problem of spectral analysis with the Dirichlet boundary conditions for the Laplace operator with a potential defined in a bounded domain of multidimensional space. The uniqueness theorem of the recovery of potential in the inverse Borg–Levinson problem with boundary conditions of the first kind is proved. A possible mathematical model of the recovery of potential is built on the basis of this theorem using incomplete spectrum.

Keywords—inverse problem; inverse problem of spectral analysis; the Laplace operator; potential; recovery of potential; the Dirichlet problem; lacunary sequence

I. INTRODUCTION

The inverse problems of spectral analysis are problems of reconstructing a linear operator by means of its given spectral characteristics (spectrums under different boundary conditions, spectral function, scattering data, etc).

The existence theorems of potential in some inverse problems of spectral analysis are known, for example, for ordinary differential equations, for powers of the Laplace operator with a potential on a rectangle. Therefore it is interesting to solve the uniqueness problem of the recovery of potential by means of given spectral characteristics in a bounded domain Ω at least from R^2 .

There is a formulation of the Borg–Levinson problem with the Dirichlet boundary conditions in H. Isozaki's paper [1]. In this paper, following H. Isozaki we consider such formulation of the Borg–Levinson problem.

In [1] H. Isozaki concluded that the lack of knowledge about the finite number of spectral data does not influence the uniqueness of the recovery of potential. Similar spectral problems were considered in [2]–[7], [10]–[17]. In some papers the authors prove the uniqueness theorems of the recovery of potential in the Borg–Levinson problem by means of incomplete spectral data of various types.

II. RESULTS AND DISCUSSION

Let Ω be a bounded domain in R^N ($N \geq 2$) with boundary $S \in C^\infty$. We consider the Dirichlet boundary value problem with function $q \in C^\infty(\overline{\Omega})$ and spectral parameter λ

$$\begin{cases} -(\Delta u)(x) + q(x)u(x) = \lambda u(x), & x \in \Omega, \\ u(x)|_S = 0. \end{cases} \quad (1)$$

The solution of this problem is a function from $C^\infty(\overline{\Omega})$ that satisfies (1) everywhere on $\overline{\Omega}$.

Let eigenvalues $\mu_1, \mu_2, \dots, \mu_K$ of problem (1) be numbered in increasing order, and let $m_i = m_i(q)$ be multiplicity of eigenvalue $\mu_i(q)$. By u_{ji} , $1 \leq j \leq m_i$, denote orthonormal eigenfunctions that correspond to eigenvalues μ_i . These sets of functions are determined ambiguously, so we consider their specific classes W_i .

We introduce the Neumann operator $N: L_2(\Omega) \rightarrow L_2(S)$ by the equality

$$N(\lambda, q)V = \frac{\partial V}{\partial \nu} \Big|_S.$$

Consider problem (1) with potentials $q_1, q_2 \in C^\infty(\overline{\Omega})$. Define scattering functions by the rule

$$F(\lambda, \omega, \theta; q_j) = \int_S (N(\lambda, q_j)\varphi_{\lambda, \omega}(x)) \overline{\varphi_{\lambda, -\theta}(x)} dS_x,$$

where S_x is a surface area element, $\varphi_{\lambda, \omega}(x) = \exp(i\sqrt{\lambda}\omega \cdot x)$, $\lambda \in C \setminus (-\infty; 0)$, $\omega, \theta \in S^{N-1}$, i is an imaginary unit, $\omega \cdot x = \sum_{k=1}^N \omega_k x_k$, $j = 1, 2$.

We know that

$$\begin{aligned} F(\lambda, \omega, \theta; q_j) &= \\ &= -\frac{\lambda}{2}(\theta - \omega)^2 \int_{\Omega} \exp(-i\sqrt{\lambda}(\theta - \omega)x) dx + \int_{\Omega} \exp(-i\sqrt{\lambda}(\theta - \omega)x) q_j(x) dx - \\ &= \int_S (-\Delta + q_j - \lambda)^{-1} (q_j \varphi_{\lambda, \omega})(x) \cdot \overline{\varphi_{\lambda, -\theta}(x)} dS_x. \end{aligned}$$

Let $\xi \in R^N$, $\xi \neq 0$ be an arbitrary fixed vector. Choose $\eta \in R^N$, $|\eta|=1$, so that η and ξ are orthogonal. Let n be a sufficiently large natural number. Define such sequences:

$$c_n = \left(1 - \frac{|\xi|^2}{4n^2}\right)^{\frac{1}{2}}, \quad \theta_n = c_n \eta + \frac{\xi}{2n}, \quad (2)$$

$$\omega_n = c_n \eta - \frac{\xi}{2n}, \quad l_n = (n+i)^2.$$

Taking into account such choice of variables l_n , θ_n , ω_n , c_n , we have

$$\lim_{n \rightarrow \infty} F(l_n, \theta_n, \omega_n; q_j) = -\frac{|\xi|^2}{2} \int_{\Omega} \exp(-ix \cdot \xi) dx + \int_{\Omega} \exp(-ix \cdot \xi) q_j(x) dx. \quad (3)$$

A certain power of Neumann operator is an integral operator. Kernel of Neumann operator is formally given by the following equality:

$$\ker N(\lambda, q_j)(x, y) = \sum_{l=1}^{\infty} \frac{\partial u_l(q_j)}{\partial v}(x) \overline{\frac{\partial u_l(q_j)}{\partial v}(y)} \cdot (\mu_l(q_j) - \lambda)^{-1},$$

where $x, y \in S$.

Showing the validity of condition

$$\|\ker N(\lambda, q_1) - \ker N(\lambda, q_2)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (4)$$

and using equality (3), we get

$$\lim_{n \rightarrow \infty} |F(l_n, \theta_n, \omega_n; q_1) - F(l_n, \theta_n, \omega_n; q_2)| = 0,$$

and therefore $q_1(x) = q_2(x)$ for any $x \in \bar{\Omega}$.

Requirement (4) is satisfied if the following condition holds:

$$\lim_{n \rightarrow \infty} \left\| \sum_{l=1}^{\infty} \frac{\frac{\partial u_l(q_1)}{\partial v}(x) \overline{\frac{\partial u_l(q_1)}{\partial v}(y)} (\mu_l(q_2) - \mu_l(q_1))}{(\mu_l(q_1) - l_n)(\mu_l(q_2) - l_n)} \right\|_{L_2(S \times S)} = 0. \quad (5)$$

Academician V.A. Ilyin and I.A. Shishmarjov showed in [8] that

$$\left\| \frac{\partial u_l}{\partial v}(q_j) \right\|_{L_2(S)} = O\left([\mu_l(q_j)]^{\frac{N}{4}}\right), \quad j=1,2.$$

Using Hörmander's theorem [9], we have

$$\mu_l \sim C_3 \cdot t^{\frac{2}{N}} \text{ as } t \rightarrow \infty.$$

Therefore, we state

$$\left\| \frac{\partial u_l}{\partial v}(q_j) \right\|_{L_2(S)} = O\left(t^{\frac{1}{2}}\right), \quad j=1,2.$$

Thus, requirement (5) is valid if

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{t_k}{|\mu_{t_k}(q_1) - l_n| |\mu_{t_k}(q_2) - l_n|} = 0. \quad (6)$$

Suppose the subsequences of eigenvalues of problems (1) satisfy condition

$$\forall k \in N \left(\frac{t_{k+1}}{t_k} \geq \lambda > 1 \right). \quad (7)$$

Such sequences are called lacunary.

In addition, consider arbitrary strictly increasing sequences of natural numbers which are satisfying the condition

$$\lim_{k \rightarrow \infty} \frac{t_{k+1} - t_k}{\sqrt{t_k}} = +\infty. \quad (8)$$

We can show that requirement (5) is true with such choice of sequence.

The replacement of natural number n by a positive real number in (2) does not disturb equality (3).

Let l_{n_k} be such that

$$\operatorname{Re} l_{n_k} = \frac{\mu_{t_{k+1}}(q_1) - \mu_{t_k}(q_1)}{2}.$$

Let us consider (6).

Fix n_s . Then $\mu_{t_s} < l_{n_s}$, and $\mu_{t_{s+1}} > l_{n_s}$.

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{t_{s+k}}{|\mu_{t_{s+k}}(q_1) - l_n| |\mu_{t_{s+k}}(q_2) - l_n|} \leq \\ & \leq |O(1)| \sum_{k=1}^{\infty} \frac{t_{s+k}}{t_{s+k}^2} = |O(1)| \sum_{k=1}^{\infty} \frac{1}{t_{s+k}} \leq |O(1)| \sum_{k=1}^{\infty} \frac{1}{\lambda^k t_s} \\ & = |O(1)| \frac{1}{t_s} \sum_{k=1}^{\infty} \frac{1}{\lambda^k} = \left| O\left(\frac{1}{t_s}\right) \right| \rightarrow 0 \end{aligned}$$

as $s \rightarrow \infty$.

Using the properties of lacunary sequences, we obtain

$$\begin{aligned} & \sum_{p=1}^s \frac{t_p}{|\mu_{t_p}(q_1) - l_n| |\mu_{t_p}(q_2) - l_n|} \leq |O(1)| \sum_{k=1}^{\infty} \frac{t_p}{\left(\frac{t_{s+1} + t_s}{2} - t_p\right)^2} \leq \\ & \leq \frac{t_s}{\left(\frac{t_s(\lambda-1)}{2}\right)^2} + \frac{t_{s-1}}{(\lambda-1)^2 \left(\frac{t_s}{2} + t_{s-1}\right)^2} + \Lambda + \\ & \quad + \frac{t_1}{(\lambda-1)^2 \left(\frac{t_s}{2} + t_{s-1} + \Lambda + t_1\right)^2} \leq \\ & \leq \frac{4t_s}{(\lambda-1)^2 t_s^2} + \frac{4t_{s-1}}{(\lambda-1)^2 (\lambda+1)^2 t_{s-1}^2} + \Lambda + \end{aligned}$$

$$\begin{aligned}
 & + \frac{4t_1}{(\lambda-1)^2(\lambda^{s-1} + \lambda^{s-2} + \Lambda + \lambda + 1)t_1^2} = \\
 & = \frac{4t_s}{(\lambda-1)^2 t_s^2} + \frac{4t_{s-1}}{(\lambda^2-1)^2 t_{s-1}^2} + \Lambda + \frac{4t_1}{(\lambda^s-1)^2 t_1^2} = \\
 & = 4 \sum_{p=1}^s \frac{1}{(\lambda^{s-p+1}-1)^2 t_p} \leq \\
 & \leq \frac{4}{(\lambda-1)^2} \sum_{p=1}^s \frac{1}{t_p} = \frac{4}{(\lambda-1)^2} O\left(\frac{1}{t_s}\right) \rightarrow 0
 \end{aligned}$$

as $s \rightarrow \infty$.

There is a similar proof for three-dimensional space. For this case we shall use the asymptotics

$$\mu_t \sim C_3 \cdot t^{\frac{2}{3}}, \quad C_3 > 0.$$

If we have N -dimensional space ($N \geq 2$), then we will require “the similarity” of problems (1) with potentials q_1 and q_2 . It ensures the uniqueness of the recovery of potential.

Let l_{n_k} be such that

$$\operatorname{Re} l_{n_k} = \frac{\mu_{t_{k+1}}(q_1) - \mu_{t_k}(q_1)}{2}.$$

Let us consider (6).

Fix n_s . Then $\mu_{t_s} < l_{n_s}$, and $\mu_{t_{s+1}} > l_{n_s}$.

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \frac{t_{s+k}^{2\alpha}}{|\mu_{t_{s+k}}(q_1) - l_{n_s}| |\mu_{t_{s+k}}(q_2) - l_{n_s}|} \leq |O(1)| \sum_{k=1}^{\infty} \frac{t_{s+k}^{2\alpha}}{t_{s+k}^{\frac{4}{N}}} = \\
 & = |O(1)| \sum_{k=1}^{\infty} \frac{1}{t_{s+k}^{\frac{4}{N}-2\alpha}} \leq |O(1)| \sum_{k=1}^{\infty} \frac{1}{\lambda^{k(\frac{4}{N}-2\alpha)} t_s^{\frac{4}{N}-2\alpha}} = \\
 & = |O(1)| \frac{1}{t_s^{\frac{4}{N}-2\alpha}} \sum_{k=1}^{\infty} \frac{1}{\lambda^{k(\frac{4}{N}-2\alpha)}} = \left| O\left(\frac{1}{t_s^{\frac{4}{N}-2\alpha}}\right) \right| \rightarrow 0
 \end{aligned}$$

as $s \rightarrow \infty$, since $\frac{4}{N} - 2\alpha > 0$, $\lambda > 1$.

Using the inequality for positive a and b

$$a + b \geq C_\gamma \cdot a^\gamma \cdot b^{1-\gamma}, \quad C_\gamma = \text{const} > 0, \quad 0 < \gamma < 1,$$

we have

$$\begin{aligned}
 & \sum_{p=1}^s \frac{t_p^{2\alpha}}{|\mu_{t_p}(q_1) - l_{n_s}| |\mu_{t_p}(q_2) - l_{n_s}|} \leq \\
 & \leq \frac{|O(1)|}{\left(t_{s+1}^{\frac{2}{N}} + t_s^{\frac{2}{N}}\right)^{1-\gamma}} \sum_{k=1}^s \frac{t_p}{\left(\frac{t_{s+1}^{\frac{2}{N}} + t_s^{\frac{2}{N}}}{2} - t_p\right)^{2\gamma}} \leq
 \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{|O(1)|}{\left(t_{s+1}^{\frac{2}{N}} + t_s^{\frac{2}{N}}\right)^{1-\gamma}} \sum_{k=1}^s \frac{1}{t_p^{\frac{4}{N}-2\alpha} (s+1-k)^2 \left(\lambda^{\frac{2}{N}} - 1\right)^2} \leq \\
 & \leq \frac{|O(1)|}{\left(t_{s+1}^{\frac{2}{N}} + t_s^{\frac{2}{N}}\right)^{1-\gamma} \left(\lambda^{\frac{2}{N}} - 1\right)^2} \cdot \sum_{p=1}^s \frac{1}{(s+1-p)^2} \leq \\
 & \leq \frac{|O(1)|}{\left(t_{s+1}^{\frac{2}{N}} + t_s^{\frac{2}{N}}\right)^{1-\gamma} \left(\lambda^{\frac{2}{N}} - 1\right)^2} \frac{s-1}{s} \leq \frac{s-1}{s^{\frac{2(1-\gamma)}{N}+1}} \rightarrow 0
 \end{aligned}$$

as $s \rightarrow \infty$.

Thus, (6) and consequently (4) hold.

III. CONCLUSION

Theorem 1. Let Ω be a bounded domain in R^N , $N = 2, 3$, with boundary $S \in C^\infty$, apart from possibly points $\{\mu_{t_k}(q_j)\}_{k=1}^\infty$, $j = 1, 2$. These points are subsequences of the eigenvalues of problems (1) with potentials q_j . Let they have property (7) or (8). If the following conditions for the orthonormal eigenfunctions of the Dirichlet problems (1) with potentials q_1 and q_2 hold:

$$a) \quad W_t(q_1) = W_t(q_2) \quad \text{as } t \neq t_k;$$

$$b) \quad \mu_t(q_1) = \mu_t(q_2) \quad \text{as } t \neq t_k,$$

then for any $x \in \bar{\Omega}$ the equality $q_1(x) = q_2(x)$ holds.

Using theorem 1, we propose a model of the recovery of potential (MRP 1).

MRP 1. Let Ω be a bounded domain in R^N , $N = 2, 3$, with boundary $S \in C^\infty$. Suppose we know all eigenvalues, apart from possibly a finite set, and values of the eigenfunctions on boundary S in problem (1) with potential q_1 . If the following conditions for problem (1) with potentials q_2 hold:

a) there exists a natural number T such that for every number $t \geq T$ we have $W_t(q_1) = W_t(q_2)$;

b) $\mu_t(q_1) = \mu_t(q_2)$ as $t \neq t_k$, where subsequences $\{\mu_{t_k}(q_2)\}_{k=1}^\infty$ of the eigenvalues of problems (1) with potentials q_2 have property (7) or (8),

then for any $x \in \bar{\Omega}$ the equality $q_1(x) = q_2(x)$ holds.

Thus, the potential q_2 in problem (1) is recovered.

Theorem 2. Let Ω be a bounded domain in R^N , $N \geq 2$, with boundary $S \in C^\infty$. Let subsequences $\{\mu_{t_k}(q_j)\}_{k=1}^\infty$, $j = 1, 2$, of the eigenvalues of problems (1) with potentials

q_j have property (7). If the following conditions for the orthonormal eigenfunctions of the Dirichlet problems (1) with potentials q_1 and q_2 hold:

a) there exists a natural number T such that for every number $t \geq T$ we have $W_t(q_1) = W_t(q_2)$;

b) $\mu_t(q_1) = \mu_t(q_2)$ as $t \neq t_k$;

c) $|\mu_{t_k}(q_1) - \mu_{t_k}(q_2)| = O(t_k^{2\alpha-1})$, where $\alpha < \frac{2}{N}$,

then for any $x \in \overline{\Omega}$ the equality $q_1(x) = q_2(x)$ holds.

Theorems 1 and 2 can be considered as the uniqueness theorems of the recovery of potential in the inverse Dirichlet problem.

Using theorem 2 we construct a model of the recovery of potential (MRP 2) similar to MRP 1.

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