Quasianalytical solution of inhomogeneous differential equation with cubic nonlinearity

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Abstract—this paper considers a method for solving Cauchy problem of nonlinear differential equation. Source of solution error and way to reduce it is studied. Solution obtained with suggested method is compared with solution obtained with built-in MATLAB functions.

Keywords—numerical methods for solving ordinary differential equations; nonlinear differential equations; Cauchy problem; phase portrait; fundamental system of solutions; equilibrium points.

I. INTRODUCTION

One of the questions in the orbit accelerators theory is studying behavior of charged particles in median plane of torus-shaped vacuum chamber. The behavior of the particles is studied both in stable and unstable oscillations region. Mathematically the question is reduced to the following equation

$$\frac{d^2 x}{d\Theta^2} + x \cdot n(x) = f(x), \quad (1)$$

where $n(x)$ is a differential characteristic of magnetic field decrease, which is created inside accelerator chamber. Function $n(x)$ is obtained by processing magnetic field measurements. It is quite properly approximated by cubic polynomial. There is a nonlinear differential equation for one kind of orbit accelerators [1]. Solving the equation is a subject for study.

$$\frac{d^2 x}{d\Theta^2} + \alpha x + \beta x^3 = \gamma x \sin(\Omega\Theta). \quad (2)$$

Modify it into

$$\frac{d^2 x}{d\Theta^2} + \omega^2(x) \cdot x = \gamma x \sin(\Omega\Theta), \quad (2)(3)$$

where

$$\omega^2(x) = \alpha + \beta x^2 \quad (4)$$

II. SOLVING HOMOGENIOUS EQUATION

The idea underlying the method is that having modified (2) into (3) and considered $\omega(x_i(\Theta_i))$ as constant for small quantity $V\Theta$ (Θ_i is left boundary of interval $\Theta_i + V\Theta$) we deduce homogeneous linear differential equation with constant

coefficients. Solution of the equation is a sum of general solution of homogeneous equation and partial solution of inhomogeneous equation. The solution of homogeneous equation is written as:

$$x(\Theta) = a \cos(\omega\Theta) + b \sin(\omega\Theta) \quad (5)$$

Let us consider in greater detail Cauchy problem solution at the first step.

We are given initial conditions (x_0, x'_0 is the derivative of x_0, Θ_0). Unknown coefficients can be found by differentiating general solution (5). Solution of Cauchy problem can be written in matrix representation.

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 \cdot \Theta_0) & \frac{-\sin(\omega_0 \cdot \Theta_0)}{\omega_0} \\ \sin(\omega_0 \cdot \Theta_0) & \frac{\cos(\omega_0 \cdot \Theta_0)}{\omega_0} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} x(\Theta_0 + \Delta\Theta) \\ x'(\Theta_0 + \Delta\Theta) \end{pmatrix} = \begin{pmatrix} \cos(\omega_0(\Theta_0 + \Delta\Theta)) & \sin(\omega_0(\Theta_0 + \Delta\Theta)) \\ -\omega_0 \sin(\omega_0(\Theta_0 + \Delta\Theta)) & \omega_0 \cos(\omega_0(\Theta_0 + \Delta\Theta)) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad (7)$$

Coordinate and its derivative values in (7) are the initial values for the solution at the next step $\Delta\Theta$. In accordance with (4) frequency is:

$$\omega^2(x(\Theta_0 + V\Theta)) = \alpha + \beta x(\Theta_0 + V\Theta)^2. \quad (8)$$

Plugging the expression for coefficients (6) in (7) we get transformation matrix for phase coordinates values from point Θ_i to point $(\Theta_i + \Delta\Theta)$.

$$\begin{pmatrix} x(\Theta_i + V\Theta) \\ x'(\Theta_i + V\Theta) \end{pmatrix} = \begin{pmatrix} \cos(\omega_i \cdot V\Theta) & \frac{\sin(\omega_i \cdot V\Theta)}{\omega_i} \\ -\omega_i \cdot \sin(\omega_i \cdot V\Theta) & \cos(\omega_i \cdot V\Theta) \end{pmatrix} \cdot \begin{pmatrix} x_i \\ x'_i \end{pmatrix} \quad (9)$$

Calculating consecutively in a loop values of phase coordinates considering the changes in $\omega(x_i(\Theta_i))$ solution of the initial equation is obtained on a discrete mesh in increments $\Delta\Theta$.

III. ACCURACY OF SOLUTION

Implementing any algorithm for solving (2) there is a question on the accuracy of the solution. This equation is noncoarse (according to the accepted in nonlinear dynamics classification, it means that the equation is particularly sensitive to computational errors and small parameter variations) and describes oscillating process in a conservative system, meaning that the amplitude is constant. Occurrence of even small errors leads to noncompliance of system's fundamental property (total energy is constant).

In order to represent properties of the solution more thoroughly we use the methods of differential equations qualitative theory. First, we find equilibrium points, then classify them and draw a phase portrait of the system in the plane (x, x') [3].

The coordinates of equilibrium points are $a - (0,0)$, $b - (-11.4, 0)$, $c - (11.4, 0)$; a is a center, b and c are saddle points. Phase portrait appears as follows.

Fig. 1. Phase portrait

Curve 1 is a potential function overlaying the phase portrait. Trajectory 2 is a separatrix, 3 is a trajectory of point a .

At the first stage of computational error estimation we compare qualitatively phase trajectories, computed by the proposed technique and by using ode45 [5] for the same initial conditions on the interval of argument change $\Theta=100*\pi$ and initial values of the phase coordinates $(11, 0)$.

For an analytical solution phase trajectory every turn is displayed at itself. Therefore, we obtain a closed parametric curve. The graph shows that the condition fails for both algorithms, the proposed one and the numerical one.

Note that computational error of the proposed one grows faster than for the numerical one.

Fig. 2. Phase trajectories in stable oscillations region

It is especially noticeable in Fig.3 a, b.

Fig. 3. Coordinates and derivatives

The proposed algorithm can obtain more accurate solution of (2) if the total energy loss is compensated at each step. Kinetic energy as a function of speed is calculated using well-known formula. Potential energy is an integral of restoring force in (2)

$$\Pi = \int (\alpha x + \beta x^3) dx = \frac{\alpha}{2} x^2 + \frac{\beta}{4} x^4. \quad (10)$$

Total energy can be found as a sum of kinetic and potential energy. Plug the initial conditions (x_0, x'_0) in expression for total energy to find W_0

$$W_0 = \frac{x_0^2}{2} + \frac{\alpha}{2} x_0^2 + \frac{\beta}{4} x_0^4. \quad (11)$$

Amplitude is decreasing follows the law that is close to exponential. So the expression for the total energy can be written as

$$W_i = e^{k \Theta} W_0. \quad (12)$$

Correction factor can be found from (12). With regard to squares of coordinates and derivatives in (11), exact values of phase coordinates are

$$\begin{aligned} x_{ti} &= \sqrt{e^{-\ln(W_i/W_0)}} \cdot x_i \\ x'_{ti} &= \sqrt{e^{-\ln(W_i/W_0)}} \cdot x'_i \end{aligned} \quad (13)$$

Corrected (9) solution (2) is displayed at Fig. 4. Initial conditions and integration interval are identical to the first phase portrait Fig. 2.

The figure shows that the points at quasianalytical solution phase trajectory are displayed at itself. It means that total energy is constant.

This is also confirmed by Fig. 5a, where oscillation amplitude remains unchanged.

Fig. 4. Corrected quasianalytical solution in stable oscillations region

Fig. 5. Coordinates and derivatives

IV. SOLUTION IN UNSTABLE OSCILLATIONS REGION

If the initial conditions are defined in a way that that mapping point at the phase plane lies outside separatrix, then according to (4) $\omega(x)^2 < 0$.

Equation (3) is written as

$$\frac{d^2 x}{d\Theta^2} - \omega^2(x) \cdot x = 0. \quad (14)$$

In this case fundamental system of solution functions are

$$\exp(-\omega\Theta); \exp(\omega\Theta), \quad (15)$$

general solution and it's derivative are written as

$$\begin{aligned} x(\Theta) &= a \exp(-\omega\Theta) + b \exp(\omega\Theta), \\ \frac{dx}{d\Theta} &= -\omega a \exp(-\omega\Theta) + \omega b \exp(\omega\Theta). \end{aligned} \quad (16)$$

Method for solving (14) is the same as method for (2). The only distinction is that trigonometric functions are replaced with exponents.

Equation (14) unlike the previous case is coarse. Properties of coarse equations are not changing a lot if parameters are changed slightly or there is a negligible error. Suggested and numerical solutions are compared without refining.

Phase portrait is plotted at Fig. 6. There are shown trajectories for four different initial conditions. Qualitatively, phase trajectories built with both methods are almost the same.

Fig. 6. Phase trajectories in unstable oscillations region

Having plugged the solutions in (14) one can see that quasianalytical solution error is a degree less than numerical solution error. That provides support to the fact the quasianalytical solution is close to analytical solution (Fig. 7).

Fig. 7. Computational error in unstable oscillation region

V. SOLVING INHOMOGENIOUS EQUATION

Partial solution of inhomogeneous differential equation without resonance ($\omega(x) \neq \Omega$) is

$$x(\Theta) = -\frac{\gamma \sin(\Omega\Theta)}{\Omega^2 - \omega^2}. \quad (17)$$

Partial solution of inhomogeneous differential equation in case of resonance ($\omega(x) = \Omega$) is

$$x(\Theta) = -\frac{\gamma(\Omega\Theta \cos(\Omega\Theta) - \sin(\Omega\Theta))}{2\Omega^2}. \quad (18)$$

Therefore general solution of (2) without resonance is

$$x(\Theta) = a \cos(\omega\Theta) + b \sin(\omega\Theta) - \frac{\gamma \sin(\Omega\Theta)}{\Omega^2 - \omega^2}, \quad (19)$$

and in case of resonance

$$x(\Theta) = a \cos(\omega\Theta) + b \sin(\omega\Theta) - \frac{\gamma(\Omega\Theta \cos(\Omega\Theta) - \sin(\Omega\Theta))}{2\Omega^2}. \quad (20)$$

Fig. 8 and Fig. 9 show corrected general solution without resonance for initial conditions (10, 0) and interval of argument change $\Theta=100*\pi$.

Fig. 8. Corrected general solution in stable oscillations region

It can be seen that phase trajectory is not displayed at itself due to influence of nonlinearity.

Fig. 9. Coordinates and derivatives

In order to perform more objective analysis for both methods we used a property of differential equation solution. Plugging the solution in differential equation reduces it to an identity.

Fig. 5 shows that Runge-Kutta method came out with three more oscillation than quasianalytical since the frequency increases as oscillation amplitude according to (4) decreases.

Furthermore, it should be noticed that the number of steps of quasianalytical method is larger than number of steps of Runge-Kutta method (Fig. 2). Corrected according to (13) quasianalytical method needs twice less steps then Runge-Kutta method.

Fig. 10. Computational error in stable oscillations region

On the base of conducted research, we can make a conclusion:

1. Designing and using of algorithms that take into account features of mathematical models of specific physical systems (in particular, total energy conservation law) let us find ways to obtain accurate solution.

2. In recent years analytical mathematical packages such as Mathematica, Maple are used more and more frequently for studying theory of differential equations and for analytical solution of specific tasks [4]. Suggested method lets enhance the abilities of those packages, because it can be implemented with built-in functions of the packages.

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