Decomposing Complete 3-Uniform Hypergraph $K^{(3)}_{34}$ into 7-cycles

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Abstract. On the basis of the definition of Hamiltonian cycle defined by Katona-Kierstead and Jianfang Wang independently. Some domestic and foreign researchers study the decomposition of complete 3-uniform hypergraph $K^{(3)}_n$ into Hamiltonian cycles and not Hamiltonian cycles. Especially, Bailey Stevens using Clique-finding the decomposition of $K^{(3)}_n$ into Hamiltonian cycles for $K^{(3)}_7$, $K^{(3)}_8$. Meszka-Rosa showed that Hamiltonian decompositions of $K^{(3)}_n$ for all admissible $n \leq 32$. Meszka-Rosa proved that a decomposition of $K^{(3)}_n$ into 5-cycles has been presented for all admissible $n \leq 17$, and for all $n = 4^m + 1$, $m$ is a positive integer. In general, the existence of a decomposition into $l(\geq 5)$-cycles remains open. The authors have given the decomposition of $K^{(3)}_n$ into 7-cycles for $n \in \{7, 8, 14, 16, 22, 23, 29, 37, 43\}$ and has showed if $K^{(3)}_n$ can be decomposition into 7-cycles, then $K^{(3)}_n$ can be decomposition into 7-cycles. In this paper, a decomposition of $K^{(3)}_{34}$ into 7-cycles is proved using the method of edge-partition and cycle sequence proposed by Jirimutu.

Introduction

Katona-Kierstead and Jianfang Wang gave the definition of Hamiltonian chain and Hamiltonian cycle in [1-2], independently. In fact, two different definitions of Hamiltonian chain and Hamiltonian cycle are the same. Some researchers studied the decomposition of complete 3-uniform hypergraph $K^{(3)}_n$ into Hamiltonian cycles and not Hamiltonian cycles in [2-9]. Especially, Bailey Stevens [3] used clique-finding the decomposition of $K^{(3)}_n$ into Hamiltonian cycles for $K^{(3)}_7$, $K^{(3)}_8$ and Meszka-Rosa [4] showed that Hamiltonian decompositions of $K^{(3)}_n$ for all admissible $n \leq 32$. Huo [10] obtained some results for all admissible $32 \leq n \leq 46$ and $n \neq 43$ using the method of edge-partition and cycle sequence. The problem of decomposing the complete 3-uniform hypergraph into 5-cycles and $l(\geq 5)$ are open. Meszka-Rosa [4] proved that a decomposition of $K^{(3)}_n$ into 5-cycles has been presented for all admissible $n \leq 17$, and for all $n = 4^m + 1$, $m$ is a positive integer. Meszka-Rosa [4] have introduced a necessary condition for the existence of 5-cycles such a decomposition is that $n \equiv 1, 2, 5, 7, 10, 11(\text{mod} 15)$. Li [11] find a decomposition of $K^{(3)}_n$ into 5-cycles for $n \in \{5, 7, 10, 11, 16, 17, 20, 22, 26\}$ and has showed if $K^{(3)}_n$ can be decomposition into 5-cycles, then $K^{(3)}_{3n}$ can be decomposed into 5-cycles. Li and Hong in [11][12] find a decomposition of $K^{(3)}_n$ into 7-cycles for $n \in \{7, 8, 14, 16, 22, 23, 29, 37, 43\}$ and Li [13] has showed if $K^{(3)}_n$ can be decomposition into 7-cycles, then $K^{(3)}_{7n}$ can be decomposition into 7-cycles. In this paper we find $K^{(3)}_{34}$ decomposed into 7-cycles.
Preliminaries

A hypergraph \( H = (V, E) \) consists of a finite set \( V \) of vertices with a family \( E \) of subsets of \( V \), called hyperedges (or simply edges). If each (hyper)edge has size \( k \), we say that \( H \) is a \( k \)-uniform hypergraph. In particular, the complete \( k \)-uniform hypergraph on \( n \) vertices has all \( k \)-subsets of \( \{0,1,\ldots,n-1\} \) as edges, denoted this by \( K_n^{(k)} \). The (hyper)edges of \( K_n^{(3)} \) is denoted by \( e(K_n^{(3)}) \).

Definition 1. Let \( H = (V, E) \) be a \( k \)-uniform hypergraph. A \( l \)-cycle in \( H \) is a cyclic sequence \( (v_0, v_1, \ldots, v_{l-1}) \) where \( 3 \leq k \leq l-1 \) such that each consecutive \( k \)-tuple of vertices is an edge of \( H \).

Definition 2. A \( l \)-cycle decomposition of \( H \) is a partition of the set of (hyper)edges of \( H \) into mutually-disjoint \( l \)-cycles.

We introduce the method of edge-partition and cycle-model in [5].

Definition 3. [4] Let \( T = \{a, b, c\} \) be a triple of distinct elements of \( \mathbb{Z}_n \). Then its difference pattern \( \pi(T) \), is the equivalence class of ordered triples containing cyclic rotations of \( (b-a, c-b, a-c) \) and \( (c-a, b-c, a-b) \) (where the differences are taken modulo \( n \)).

Clearly, the three differences sum to zero. Therefore if we know that the first two differences are \( x \) and \( y \), then the third is \( n-x-y \). Omit the third number, we can get a difference pair. Using edge-partition of \( K_n^{(3)} \) in paper [5], all difference pairs of hypergraphs \( K_n^{(3)} \) may be obtained. Let \( Z \) be the set of integers, \( n \) be a fixed positive integer, and \( \mathbb{Z}_n = \{0,1,\ldots,n-1\} \). Let

\[
D_{all}(n) = \{ (k_1, k_2) \mid 1 \leq k_1, k_2 \leq n-1, \text{ and } k_1 + k_2 \neq n \}
\]

\[
D(n) = D_\ell(n) \cup D_k(n) \cup D_m(n)
\]

where

\[
D_\ell(n) = \left\{ (k_1, k_2) \in D_{all}(n) \mid k_1 = k_2 = k, \quad \exists 1 \leq k < \frac{n}{2} \right\}
\]

\[
D_k(n) = \left\{ (k_1, k_2) \in D_{all}(n) \mid 1 \leq k_1 < k_2 < \frac{n-k_1}{2} \right\}
\]

\[
D_m(n) = \left\{ (k_2, k_1) \in D_{all}(n) \mid (k_1, k_2) \in D_\ell(n) \right\}.
\]

Given a difference pair \( (k_1, k_2) \in D_{all}(n) \) and an integer \( m \in \mathbb{Z}_n \), define a subhypergraph of \( K_n^{(3)} \) generated by \( (k_1, k_2) \) as follows:

\[
E(m; k_1, k_2) = \{ m, m+k_1, m+k_1+k_2 \} \pmod{n}
\]

Denoted by

\[
H(k_1, k_2) = \{E(m; k_1, k_2) \mid m \in \mathbb{Z}_n \}
\]

where the addition is performed modulo \( n \).

Lemma 1. [5] Let \( (k_1, k_2) \) and \( (k'_1, k'_2) \) be arbitrary two distinct difference pairs in \( D_{all}(n) \), we have either \( H(k_1, k_2) \cap H(k'_1, k'_2) = \emptyset \) or \( H(k_1, k_2) = H(k'_1, k'_2) \), and a necessary and sufficient condition for the second equation is
\[(k_1, k_2) \equiv \begin{cases} 
(k_1', k_2') & \text{or} \\
(k_1' + k_2', -k_2') & \text{or} \\
(-k_1', k_1' + k_2') & \text{or} \\
(k_2', -k_1' - k_2') & \text{or} \\
(-k_1' - k_2', k_1') & \text{or} \\
(-k_1', -k_2') & \text{or} \\
(k_1, k_2) & \text{or} 
\end{cases} \pmod{n}.
\]

**Definition 4**[5]. Let \((k_1, k_2)\) and \((k_1', k_2')\) be arbitrary two distinct difference pairs in \(D_{all}(n)\). We say \((k_1, k_2)\) and \((k_1', k_2')\) are equivalent if \(H(k_1, k_2) = H(k_1', k_2')\). This is denoted by \((k_1, k_2) \sim (k_1', k_2')\).

**Theorem 2**[5]. (Edge-partition of \(K_n^{(3)}\)) For any \(K_n^{(3)}\),
\[
\mathcal{E}(K_n^{(3)}) = \bigcup_{(k_1, k_2) \in D(n)} H(k_1, k_2),
\]
where \((k_1, k_2) \in D(n)\). If \(k_1 \neq k_2\), for convenience, we use \((k_1, k_2)\) denote \((k_1, k_2)\) and \((k_2, k_1)\).

**Definition 5.** Let \(n\) be a positive integer, for any \(0 \leq i, j \leq l - 1\), \((k_i, k_{i+1}) \in D_{all}(n)\), \((k_i, k_{i+1})\) and \((k_j, k_{j+1})\) are inequitable when \(i \neq j\), obtain that \((k_0, k_1, \ldots, k_{l-1})\) be a sequence on \(D_{all}(n)\). The sequence \((k_0, k_1, \ldots, k_{l-1})\) induces the cycle sequence
\[
(r_0, r_1, \ldots, r_{l-1}) \quad (1)
\]
Sequence Eq.1 satisfies the following two conditions:

\(a\). \(r_0 = 0, \sum_{i=0}^{l} k_i \equiv r_j \pmod{n}, r_j = r_0 = 0.\)

\(b\). For any \(i, j (i \neq j)\), \(r_i \neq r_j\).

Then \((r_0, r_1, \ldots, r_{l-1})\) is a \(l\)-cycle, denoted by \(C_l = (r_0, r_1, \ldots, r_{l-1})\), called base cycle. According to the difference pattern \(\pi(T)\), obviously, we obtained the set of \(l\)-cycles \(\{C_i + i \mid i \in \mathbb{Z}_n\}\), where \(C_i + i = (r_0 + i, r_1 + i, \ldots, r_{l-1} + i) \pmod{n} \). In particular, if \(l = n\), \((r_0, r_1, \ldots, r_{l-1})\) is a base Hamiltonian cycle, denoted by \(C_n = (r_0, r_1, \ldots, r_{n-1})\).

**Lemma 3.** Let \(n\) be a positive integer, for any \(0 \leq i, j \leq l - 1\), \((k_i, k_{i+1}) \in D_{all}(n)\), \((k_i, k_{i+1})\) and \((k_j, k_{j+1})\) are inequitable when \(i \neq j\), obtain that \((k_0, k_1, \ldots, k_{l-1})\) be a sequence on \(D_{all}(n)\).

Then
\[
H(k_0, k_1, \ldots, k_{l-1}) = \bigcup_{i=0}^{l-1} H(k_i, k_{i+1}),
\]
where \(k_0 = k_0\).

**Decomposing \(K_n^{(3)}\) Into 7-cycles**

**Theorem 1.** \(K_n^{(3)}\) can be decomposed into 7-cycles.
Proof. We can decompose the edges of \( K_{44}^{(3)} \) into 1892 7-cycles produced by 43-base 7-cycles as follows: we have since \( |e(K_{44}^{(3)})| = \binom{44}{7} = 13244 \) edges and 7 \( \mid 13244 \), we have

\[
D(44) = \{ (1,1),(2,2),(3,3),(4,4),(5,5), (6,6),(7,7),(8,8),(9,9),(10,10),(11,11),(12,12), (13,13),(14,14),(15,15),(16,16),(17,17),(18,18),(19,19),(20,20),(21,21),(22,22), \ldots \}
\]

Now, we need to find the decomposition of \( K_{44}^{(3)} \). On \( D(44) \), according to Definition 5, we obtain 43 sequences as follows:

\[
\begin{align*}
(1) & \quad (1,1,2,1,3,1,35) \\
(2) & \quad (1,4,42,5,38,5,37) \\
(3) & \quad (1,7,40,42,8,2,32) \\
(4) & \quad (1,9,37,4,1,10,26) \\
(5) & \quad (1,12,34,2,9,1,29) \\
(6) & \quad (1,13,33,3,2,12,24) \\
(7) & \quad (1,15,31,3,12,23) \\
(8) & \quad (1,16,30,3,5,8,25) \\
(9) & \quad (1,21,25,2,15,3,21) \\
(10) & \quad (1,23,23,3,6,13,19) \\
(11) & \quad (1,25,21,3,7,14,17) \\
(12) & \quad (1,27,19,3,9,14,15) \\
(13) & \quad (1,30,16,2,16,11,12) \\
(14) & \quad (1,32,14,3,13,15,10) \\
(15) & \quad (1,36,11,2,20,16,6) \\
(16) & \quad (2,20,27,40,6,17,20) \\
(17) & \quad (2,24,24,39,7,21,15) \\
(18) & \quad (2,29,19,39,8,24,11) \\
(19) & \quad (2,35,12,40,8,29,6) \\
(20) & \quad (3,14,34,38,9,8,26) \\
(21) & \quad (3,16,32,8,30,10,11,22) \\
(22) & \quad (3,23,25,38,11,15,17) \\
(23) & \quad (3,25,23,38,12,18,13) \\
(24) & \quad (3,29,20,37,11,25,7) \\
(25) & \quad (3,30,18,38,13,20,10) \\
(26) & \quad (4,5,4,6,30,6,33) \\
(27) & \quad (4,8,5,28,5,14,24) \\
(28) & \quad (4,9,4,28,7,27,29) \\
(29) & \quad (4,13,38,34,5,16,22) \\
(30) & \quad (4,14,35,36,16,9,18) \\
(31) & \quad (4,15,34,37,12,14,16) \\
(32) & \quad (4,23,26,36,21,12,10) \\
(33) & \quad (4,27,22,37,22,9,11) \\
(34) & \quad (5,7,11,27,10,10,18) \\
(35) & \quad (5,12,5,23,11,19,13) \\
(36) & \quad (5,13,6,23,29,34,22) \\
(37) & \quad (5,19,9,15,13,17,10) \\
(38) & \quad (5,30,24,14,9,26,24) \\
(39) & \quad (6,15,9,25,12,12,9) \\
(40) & \quad (6,27,36,22,14,7,20) \\
(41) & \quad (7,13,10,16,20,13,9) \\
(42) & \quad (7,16,12,17,11,7,18) \\
(43) & \quad (8,19,11,18,9,10,13)
\end{align*}
\]

Let \( D \) is a collection of the 43 sequences above. Thus they correspond to 43-base 7-cycles:
By the method of edge-partition, we obtain the decomposition of $K_{44}^{(3)}$ into 1892 7-cycles, that is

$$
\mathcal{E}(K_{44}^{(3)}) = \bigcup_{(k_1, k_2) \in D(44)} H(k_1, k_2) = \bigcup_{(k_i, k_i, \ldots, k_6) \in D_{44}(44)} H(k_1, k_2, \ldots, k_6)
$$

$$
= \bigcup_{j=1}^{43} \left\{ C_{7(i)} + j, j \in \mathbb{Z}_{44} \right\}
$$

where $C_{7(i)} = (r_0, r_1, \ldots, r_6)$, $C_{7(i)} + j = \{r_0 + j, r_1 + j, \ldots, r_6 + j\}$ (mod 44).

Hence we obtain the decomposition of $K_{44}^{(3)}$ into 1892 7-cycles.

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References