Decomposing Complete 3-Uniform Hypergraph $K_{44}^{(3)}$ into 7-cycles

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Abstract. On the basic of the definition of Hamiltonian cycle defined by Katona-Kierstead and Jianfang Wang independently. Some domestic and foreign researchers study the decomposition of complete 3-uniform hypergraph $K_n^{(3)}$ into Hamiltonian cycles and not Hamiltonian cycles. Especially , Bailey Stevens using Clique - finding the decomposition of $K_n^{(3)}$ into Hamiltonian cycles for $K_7^{(3)}$, $K_8^{(3)}$. Meszka-Rosa showed that Hamiltonian decompositions of $K_n^{(3)}$ for all admissible $n \le 32$. Meszka-Rosa proved that a decomposition of $K_n^{(3)}$ into 5-cycles has been presented for all admissible $n \le 17$, and for all $n = 4^m + 1$, m is a positive integer. In general, the existence of a decomposition into $l(\ge 5)$ -cycles remains open. The authors have given the decomposition of $K_n^{(3)}$ into 7-cycles for $n \in \{7,8,14,16,22,23,29,37,43\}$ and has showed if $K_n^{(3)}$ can be decomposition into 7-cycles, then $K_{7n}^{(3)}$ can be decomposition into 7-cycles. In this paper, a decomposition of $K_{44}^{(3)}$ into 7-cycles is proved using the method of edge-partition and cycle sequence proposed by Jirimutu.

Introduction

Katona-Kierstead and Jianfang Wang gave the definition of Hamiltonian chain and Hamiltonian cycle in [1-2], independently. In fact, two different definitions of Hamiltonian chain and Hamiltonian cycle are the same. Some researchers studied the decomposition of complete 3-uniform hypergraph $K_n^{(3)}$ into Hamiltonian cycles and not Hamiltonian cycles in [2-9]. Especially , Bailey Stevens [3] used clique-finding the decomposition of $K_n^{(3)}$ into Hamiltonian cycles for $K_7^{(3)}$, $K_8^{(3)}$ and Meszka-Rosa [4] showed that Hamiltonian decompositions of $K_n^{(3)}$ for all admissible $n \le 32$. Huo [10] obtained some results for all admissible $32 \le n \le 46$ and $n \ne 43$ using the method of edge-partition and cycle sequence. The problem of decomposing the complete 3-uniform hypergraph into 5-cycles and $l(>5)$ are open. Meszka-Rosa [4] proved that a decomposition of $K_n^{(3)}$ into 5-cycles has been presented for all admissible $n \le 17$, and for all $n = 4^m + 1$, m is a positive integer. Meszka-Rosa [4] have introduced a necessary condition for the existence of 5-cycles such a decomposition is that $n \equiv 1,2,5,7,10\,,11 \pmod{15}$. Li [11] find a decomposition of $K_n^{(3)}$ into 5-cycles for $n \in \{5,7,10,11,16,17,20,22,26\}$ and has showed if $K_n^{(3)}$ can be decomposition into 5-cycles, then $K_{5n}^{(3)}$ can be decomposed into 5-cycles. Li and Hong in [11][12] find a decomposition of $K_n^{(3)}$ into 7-cycles for $n \in \{7,8,14,16,22,23,29,37,43\}$ and Li [13] has showed if $K_n^{(3)}$ can be decomposition into 7-cycles, then $K_{7n}^{(3)}$ can be decomposition into 7-cycles. In this paper we find $K_{44}^{(3)}$ decomposed into 7- cycles.

Preliminaries

A hypergraph $H=(V,E)$ consists of a finite set V of vertices with a family E of subsets of V, called hyperedges (or simply edges). If each (hyper)edge has size k, we say that H is a k-uniform hypergraph. In particular, the complete k-uniform hypergraph on n vertices has all k-subsets of $Z_n=\{0,1,\cdots,n-1\}$ as edges, denoted this by $K_n^{(k)}$. The (hyper)edges of $K_n^{(3)}$ is denoted by $\varepsilon\left(K_n^{(3)}\right)$.

Definition 1. Let $H=(V,E)$ be a k-uniform hypergraph. A l-cycle in H is a cyclic sequence k of $(v_0,v_1,\cdots,v_{l-1})$ where $3\le k\le l-1$ such that each consecutive k -tuple of vertices is an edge of H.

Definition 2. A l-cycle decomposition of H is a partition of the set of (hyper)edges of H into mutually-disjoint l-cycles.

We introduce the method of edge-partition and cycle-model in [5].

Definition 3.[4] Let $T=\{a,b,c\}$ be a triple of distinct elements of Z_n. Then its difference pattern $\pi(T)$, is the equivalence class of ordered triples containing cyclic rotations of $(b-a,c-b,a-c)$ and $(c-a,b-c,a-b)$ (where the differences are taken modulo n).

Clearly, the three differences sum to zero. Therefore if we know that the first two differences are x and y, then the third is $n-x-y$. Omit the third number, we can get a difference pair. Using edge-partition of $K_n^{(3)}$ in paper [5], all difference pairs of hypergraphs $K_n^{(3)}$ may be obtained. Let Z be the set of integers, n be a fixed positive integer, and $Z_n=\{0,1,\cdots,n-1\}$. Let

$$D_{all}(n)=\{\ (k_1,k_2)\,|\,1\le k_1,k_2\le n-1,\ and\ k_1+k_2\ne n\ \}$$

$$D(n)=D_e(n)\cup D_l(n)\cup D_m(n)$$

where

$$D_e(n)=\left\{\ (k_1,k_2)\in D_{all}(n)\ \middle|\ k_1=k_2=k,\ 且\ 1\le k<\frac{n}{2}\ \right\}$$

$$D_l(n)=\left\{\ (k_1,k_2)\in D_{all}(n)\ \middle|\ 1\le k_1<k_2<\frac{n-k_1}{2}\ \right\}$$

$$D_m(n)=\{\ (k_2,k_1)\in D_{all}(n)\ |\ (k_1,k_2)\in D_l(n)\ \}.$$

Given a difference pair $(k_1,k_2)\in D_{all}(n)$ and an integer $m\in Z_n$, define a subhypergraph of $K_n^{(3)}$ generated by (k_1,k_2) as follows:

$$E(m;k_1,k_2)\equiv\{\ m,\ m+k_1,\ m+k_1+k_2\ \}\pmod n$$

Denoted by

$$H(k_1,k_2)=\{E(m;k_1,k_2)\ |\ m\in Z_n\}$$

where the addition is performed modulo n.

Lemma 1.[5] Let (k_1,k_2) and (k_1',k_2') be arbitrary two distinct difference pairs in $D_{all}(n)$, we have either $H(k_1,k_2)\cap H(k_1',k_2')=\varnothing$ or $H(k_1,k_2)=H(k_1',k_2')$, and a necessary and sufficient condition for the second equation is

$$(k_1,k_2)\equiv\begin{cases}(k_1',\ k_2') & or\\ (k_1'+k_2',-k_2') & or\\ (-k_1',\ k_1'+k_2') & or\\ (k_2',-k_1'-k_2') & or\\ (-k_1'-k_2',\ k_1') & or\\ (-k_2',-k_1') & \end{cases}\pmod n.$$

Definition 4[5]. Let (k_1,k_2) and (k_1',k_2') be arbitrary two distinct difference pairs in $D_{all}(n)$. We say (k_1,k_2) and (k_1',k_2') are equivalent if $H(k_1,k_2)=H(k_1',k_2')$. This is denoted by $(k_1,\ k_2)\sim(k_1',k_2')$.

Theorem 2[5]. (Edge-partition of $K_n^{(3)}$)For any $K_n^{(3)}$,

$$\varepsilon\left(K_n^{(3)}\right)=\bigcup_{(k_1,k_2)\in D(n)}H(k_1,k_2),$$

where $(k_1,k_2)\in D(n)$. If $k_1\neq k_2$, for convenience, we use (k_1,k_2) denote (k_1,k_2) and (k_2,k_1).

Definition 5. Let n be a positive integer, for any $0\le i,j\le l-1$, $(k_i,k_{i+1})\in D_{all}(n)$, (k_i,k_{i+1}) and (k_j,k_{j+1}) are inequitable when $i\neq j$, obtain that $(k_0,\ k_1,\ \cdots,\ k_{l-1})$ be a sequence on $D_{all}(n)$.The sequence $(k_0,\ k_1,\ \cdots,\ k_{l-1})$ induces the cycle sequence

$$\left(r_0,\ r_1,\ \cdots,\ r_{l-1}\right)\tag{1}$$

Sequence Eq.1 satisfies the following two conditions:

$a)$. $r_0=0,\sum_{i=0}^{j}k_i\equiv r_j\pmod n, r_l=r_0=0$.

$b)$. For any $i,j\ (i\neq j), r_i\neq r_j$.

Then $(r_0,\ r_1,\ \cdots,\ r_{l-1})$ is a l-cycle, denoted by $C_l=(r_0,\ r_1,\ \cdots,\ r_{l-1})$, called base cycle. According to the difference pattern $\pi(T)$, obviously, we obtained the set of l-cycles $\{C_l+i\,|\,i\in Z_n\}$, where , $C_l+i=(r_0+i,\ r_1+i,\ \cdots,\ r_{l-1}+i)(\mathrm{mod}\,n)$. In particular, if $l=n$, $(r_0,\ r_1,\ \cdots,\ r_{l-1})$ is a base Hamiltonian cycle, denoted by $C_n=(r_0,\ r_1,\ \cdots,\ r_{n-1})$.

Lemma 3. Let n be a positive integer, for any $0\le i,j\le l-1$, $(k_i,k_{i+1})\in D_{all}(n)$, (k_i,k_{i+1}) and (k_j,k_{j+1}) are inequitable when $i\neq j$, obtain that $(k_0,\ k_1,\ \cdots,\ k_{l-1})$ be a sequence on $D_{all}(n)$.

Then

$$H(k_0,\ k_1,\ \cdots,\ k_{l-1})=\bigcup_{i=0}^{l-1}H(k_i,\ k_{i+1}),$$

where $k_l=k_0$.

Decomposing $K_{44}^{(3)}$ Into 7-cycles

Theorem 1. $K_{44}^{(3)}$ can be decomposed into 7-cycles.

Proof. We can decompose the edges of $K_{44}^{(3)}$ into 1892 7-cycles produced by 43-base 7-cycles as follows: we have since $\left|\varepsilon\left(K_{44}^{(3)}\right)\right| = \binom{44}{3} = 13244$ edges and 7 | 13244, we have

$$\begin{aligned} D(44) = \{&(1,1),(2,2),(3,3),(4,4),(5,5),\ (6,6),(7,7),(8,8),(9,9),(10,10),(11,11),(12,12),\\ &(13,13),(14,14),(15,15),(16,16),(17,17),(18,18),(19,19),(20,20),(21,21),(1,2),\\ &(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(1,11),(1,12),(1,13),(1,14),(1,15),\\ &(1,16),(1,17),(1,18),(1,19),(1,20),(1,21),(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),\\ &(2,9),(2,10),(2,11),(2,12),(2,13),(2,14),(2,15),(2,16),(2,17),(2,18),(2,19),\\ &(2,20),(3,4),(3,5),(3,6),(3,7),(3,8),(3,9),(3,10),(3,11),(3,12),(3,13),(3,14),\\ &(3,15),(3,16),(3,17),(3,18),(3,19),(3,20),(4,5),(4,6),(4,7),(4,8),(4,9),(4,10),\\ &(4,11),(4,12),(4,13),(4,14),(4,15),(4,16),(4,17),(4,18),(4,19),(5,6),(5,7),(5,8),\\ &(5,9),(5,10),(5,11),(5,12),(5,13),(5,14),(5,15),(5,16),(5,17),(5,18),(5,19),(6,7),\\ &(6,8),(6,9),(6,10),(6,11),(6,12),(6,13),(6,14),(6,15),(6,16),(6,17),(6,18),(7,8),\\ &(7,9),(7,10),(7,11),(7,12),(7,13),(7,14),(7,15),(7,16),(7,17),(7,18),(8,9),(8,10),\\ &(8,11),(8,12),(8,13),(8,14),(8,15),(8,16),(8,17),(9,10),(9,11),(9,12),(9,13),(9,14),\\ &(9,15),(9,16),(9,17),(10,11),(10,12),(10,13),(10,14),(10,15),(10,16),(11,12),\\ &(11,13),(11,14),(11,15),(11,16),(12,13),(12,14),(12,15),(13,14),(13,15).\} \end{aligned}$$

Now, we need to find the decomposition of $K_{44}^{(3)}$. On $D(44)$, according to Definition 5, we obtain 43 sequences as follows:

$$\begin{array}{lll} (1)(1,1,2,1,3,1,35) & (2)(1,4,42,5,38,5,37) & (3)(1,7,40,42,8,2,32) \\ (4)(1,9,37,4,1,10,26) & (5)(1,12,34,2,9,1,29) & (6)(1,13,33,3,2,12,24) \\ (7)(1,15,31,3,3,12,23) & (8)(1,16,30,3,5,8,25) & (9)(1,21,25,2,15,3,21) \\ (10)(1,23,23,3,6,13,19) & (11)(1,25,21,3,7,14,17) & (12)(1,27,19,3,9,14,15) \\ (13)(1,30,16,2,16,11,12) & (14)(1,32,14,3,13,15,10) & (15)(1,36,11,42,20,16,6) \\ (16)(2,20,27,40,6,17,20) & (17)(2,24,24,39,7,21,15) & (18)(2,29,19,39,8,24,11) \\ (19)(2,35,12,40,8,29,6) & (20)(3,14,34,38,9,8,26) & (21)(3,16,32,38,10,11,22) \\ (22)(3,23,25,38,11,15,17) & (23)(3,25,23,38,12,18,13) & (24)(3,29,20,37,11,25,7) \\ (25)(3,30,18,38,13,20,10) & (26)(4,5,4,6,30,6,33) & (27)(4,8,5,28,5,14,24) \\ (28)(4,9,4,28,7,7,29) & (29)(4,13,38,34,5,16,22) & (30)(4,14,35,36,16,9,18) \\ (31)(4,15,34,37,12,14,16) & (32)(4,23,26,36,21,12,10) & (33)(4,27,22,37,22,9,11) \\ (34)(5,7,11,27,10,10,18) & (35)(5,12,5,23,11,19,13) & (36)(5,13,6,23,29,34,22) \\ (37)(5,19,9,15,13,17,10) & (38)(5,30,24,14,9,26,24) & (39)(6,15,9,25,12,12,9) \\ (40)(6,27,36,22,14,7,20) & (41)(7,13,10,16,20,13,9) & (42)(7,16,12,17,11,7,18) \\ (43)(8,19,11,18,9,10,13) & & \end{array}$$

Let D is a collection of the 43 sequences above. Thus they correspond to 43-base 7-cycles:

$C_{7(1)}=(0,1,2,4,5,8,9)$ $C_{7(2)}=(0,1,5,3,8,2,7)$ $C_{7(3)}=(0,1,8,4,2,10,12)$
$C_{7(4)}=(0,1,10,3,7,8,18)$ $C_{7(5)}=(0,1,13,3,5,14,15)$ $C_{7(6)}=(0,1,14,3,6,8,20)$
$C_{7(7)}=(0,1,16,3,6,9,21)$ $C_{7(8)}=(0,1,17,3,6,11,19)$ $C_{7(9)}=(0,1,22,3,5,20,23)$
$C_{7(10)}=(0,1,24,3,6,12,25)$ $C_{7(11)}=(0,1,26,3,6,13,27)$ $C_{7(12)}=(0,1,28,3,6,15,29)$
$C_{7(13)}=(0,1,31,3,5,21,32)$ $C_{7(14)}=(0,1,33,3,6,19,34)$ $C_{7(15)}=(0,1,37,4,2,22,38)$
$C_{7(16)}=(0,2,22,5,1,7,24)$ $C_{7(17)}=(0,2,26,6,1,8,29)$ $C_{7(18)}=(0,2,31,6,1,9,33)$
$C_{7(19)}=(0,2,37,5,1,9,38)$ $C_{7(20)}=(0,3,17,7,1,10,18)$ $C_{7(21)}=(0,3,19,7,1,11,22)$
$C_{7(22)}=(0,3,26,7,1,12,27)$ $C_{7(23)}=(0,3,28,7,1,13,31)$ $C_{7(24)}=(0,3,32,8,1,12,37)$
$C_{7(25)}=(0,3,33,7,1,14,34)$ $C_{7(26)}=(0,4,9,13,19,5,11)$ $C_{7(27)}=(0,4,12,17,1,6,20)$
$C_{7(28)}=(0,4,13,17,1,8,15)$ $C_{7(29)}=(0,4,17,11,1,6,22)$ $C_{7(30)}=(0,4,18,9,1,17,26)$
$C_{7(31)}=(0,4,19,9,2,14,28)$ $C_{7(32)}=(0,4,27,9,1,22,34)$ $C_{7(33)}=(0,4,31,9,2,24,33)$
$C_{7(34)}=(0,5,12,23,6,16,26)$ $C_{7(35)}=(0,5,17,22,1,12,31)$ $C_{7(36)}=(0,5,18,24,3,32,22)$
$C_{7(37)}=(0,5,24,33,4,17,34)$ $C_{7(38)}=(0,5,35,15,29,38,20)$ $C_{7(39)}=(0,6,21,30,11,23,35)$
$C_{7(40)}=(0,6,33,25,3,17,24)$ $C_{7(41)}=(0,7,20,30,2,22,35)$ $C_{7(42)}=(0,7,23,35,8,19,26)$
$C_{7(43)}=(0,8,27,38,12,21,31)$

By the method of edge-partition, we obtain the decomposition of $K_{44}^{(3)}$ into 1892 7-cycles, that is

$$\varepsilon\left(K_{44}^{(3)}\right)=\bigcup_{(k_1,k_2)\in D(44)} H(k_1,k_2)=\bigcup_{(k_{i0},k_{i1}\cdots k_{i6})\in D_{all}(44)} H(k_{i0},k_{i1},\cdots,k_{i6})$$
$$=\bigcup_{i=1}^{43}\left\{C_{7(i)}+j,\ j\in Z_{44}\right\}$$

where $C_{7(i)}=(r_{i0},\ r_{i1},\ \cdots,\ r_{i6})$, $C_{7(i)}+j=\{r_{i0}+j,\ r_{i1}+j,\ \cdots,r_{i6}+j\}(\bmod 44)$.

Hence we obtain the decomposition of $K_{44}^{(3)}$ into 1892 7-cycles.

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