Optimal Portfolio Choice under Hidden Regime Switching Model

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Abstract

We investigate a portfolio optimization problem in a continuous-time Markov-modulated financial market. The unobservable mean return of a risky asset follows a continuous-time, two-state Markov chain whose states are interpreted as different states of market. Using results from filter theory, we reduce this problem to one with complete observation. We solve the problem by stochastic control methods and the optimal portfolio can be explicitly characterized by stochastic integrals. The Monte Carlo simulations are implemented to compute the optimal portfolio allocations. The results show that state uncertainty have a great influence on optimal portfolio choice. The parameter uncertainty prompts the investor to hedge against unanticipated changes in the state variables.

Keywords

Portfolio choice; hidden regime switching; stochastic control methods; Monte Carlo simulation

1. Introduction

The traditional portfolio theory assumes that the asset returns are generated by a linear process with stable parameters. However, a large number of empirical researches have shown that returns often show a nonlinear and dynamic structural change. The financial market contains bull market and bear market. In different market states, the return and risk of assets are asymmetric (Longin et al. ^[1] and Ang et al. ^[2]).

Recently, many studies have turned to the optimal portfolio selection problem under Markovian regime-switching model. Empirically, the Markov regime switching model provide better fit to many economic and financial time series compared with their non-regime-switching counterparts. Ang et al. ^[3-4] and Liow et al. ^[5] consider the international

asset allocation problem under an observable Markov-modulated drift in a discrete-time setting. They show that the optimal investment weight depends on a variety of market states and investment horizon, and the regime-switching strategy dominated static strategies in an out-of-sample test. Guidolin et al. ^[6-7] promote the model of Ang et al. ^[3-4] and assume regimes as unobservable. They find that optimal asset allocations vary considerably across these states and change over time as investors revise their estimates of the state probabilities. Guidolin ^[8] consider international asset allocation under skew and kurtosis preferences with the presence of regimes in the return distribution. Tu ^[9] proposes a Bayesian framework for the decision problem under a regime-switching model. However, the studies mentioned above are discrete time models. Honda ^[10] uses hidden Markov chain to describe the unobservable market state. He solves a

continuous-time portfolio and consumption problem in which the expected return of a single risky asset depends on a two states hidden Markov chain. Optimal policies are computed using Monte Carlo methods. The optimal solution indicates that fluctuations in the estimated mean returns will cause hedging demand of assets. Furthermore, Liu ^[11] studies the optimal portfolio decisions in continuous time under ambiguity for regime switching mean returns for a long-horizon investor.

In this paper, we extend the model of Guidolin ^[6-7] to study a continuous-time portfolio optimization problem in which the drift rate of the stock is two states hidden Markov modulated. An explicit solution is obtained from the stochastic dynamic programming principle. Our paper shares a similar set up but extends Honda's results in several directions by investigating explicit solution with stochastic control method, by entertaining economic implications of the model at the time focusing on the mathematical model, by performing Monte Carlo simulations to compute the optimal portfolios. We also solve the problem explicitly with all risk aversion coefficient, while Honda ^[10] only showed the general solution to the case that the risk aversion coefficient is 0.5.

2. The Basic Model

Let (Ω, F, P) be a probability space on which a standard Brownian motion W_t and a two-state, continuous-time Markov chain Y_t are defined. One risky and one riskless asset are available for investment. The return on riskless asset is a positive constant r . The risky asset price process S_t satisfies

$$dS_t = S_t \mu(Y_t) dt + S_t \sigma dW_t \quad (1)$$

Where σ is a constant. The expected return $\mu(Y_t)$ on the risky asset depends on a state variable Y_t . The state space is $\{0, 1\}$,

while $\mu_H = \mu(1), \mu_L = \mu(0)$,

$\mu_H > \mu_L$, meaning that state 1 is the bull market with a high expected return and state 0 is the bear market with a low expected return. The infinitesimal generating matrix of Y_t is $\begin{pmatrix} -\lambda & \lambda \\ \nu & -\nu \end{pmatrix}$,

where $\lambda, \nu > 0$.

Define $\pi_t = P(Y_t = 1 | F_t^S)$ ($\pi_0 = x$).

π_t is known as the filtered probability which is a prior probability that the current regime is the high expected return state. According to the definition of Y_t , we can see that $\pi_t = E[Y_t | F_t^S]$.

So π_t reflects the investor's expectations for the market state.

According to Theorem 9.1 of Liptser et al ^[12], π_t follows the stochastic differential equation

$$d\pi_t = [\lambda - (\lambda + \nu)\pi_t] dt + \pi_t(1 - \pi_t) \frac{\mu_H - \mu_L}{\sigma} d\hat{W}_t \quad (2)$$

Where $\hat{W}_t$ is the standard Brownian motion with respect to $\{F_t^S\}$ defined by

$$d\hat{W}_t = \frac{dS_t - S_t \hat{\mu}_t dt}{S_t \sigma} \quad (3)$$

Where

$\hat{\mu}_t = E[\mu_t | F_t^S] = \pi_t \mu_H + (1 - \pi_t) \mu_L$, then the process satisfies

$$d\pi_t = [\lambda - (\lambda + \nu)\pi_t + \frac{\pi_t(\mu_H - \hat{\mu}_t)(\mu_L - \hat{\mu}_t)}{\sigma^2}] dt + \frac{\pi_t(\mu_L - \hat{\mu}_t)}{\sigma} dW_t \quad (4)$$

3. Optimal portfolio choice model

Time is continuous in the finite horizon $[t,$

T]. The utility function is the CRRA utility function. X_t is the wealth process, γ is the risk aversion coefficient. A portfolio choice b_t is a $\{F_t^S\}$ adapted progressively measurable process such that $\int_0^T |b_s|^2 ds < \infty$. The budget constraint is given by

$$dX_t = [r + b_t(\mu_t - r)]X_t dt + b_t \sigma X_t dW_t \quad (5)$$

Which can be rewritten as

$$dX_t = [r + b_t(\hat{\mu}_t - r)]X_t dt + b_t \sigma X_t d\hat{W}_t \quad (6)$$

Based on the information set $\{F_t^S\}$, the investor makes portfolio decisions to maximize the expected lifetime utility

$$\max_b E[U(X_T) | F_t^S]$$

$$st \ dX_t = [r + b_t(\mu_t - r)]X_t dt + b_t \sigma X_t dW_t \quad (7)$$

The optimal problem can be rewritten as follow

$$\max_b E[U(X_T) | F_t^S]$$

$$st \ dX_t = [r + b_t(\hat{\mu}_t - r)]X_t dt + b_t \sigma X_t d\hat{W}_t$$

$$d\pi_t = [\lambda - (\lambda + \nu)\pi_t]dt + \pi_t(1 - \pi_t) \frac{\mu_H - \mu_L}{\sigma} d\hat{W}_t \quad (8)$$

Since $\hat{W}_t, \hat{\mu}_t, \pi_t$ are $\{F_t^S\}$ adapted, so we transform the incomplete information model into complete information model. Then, the standard methods can be used to solve the adjusted optimization problem

Define the value function $J^{(b)}(t, X, \Pi) @ \max_b E[U(X_T) | F_t^S]$. From the stochastic optimal control theory, the value function satisfies the following HJB partial differential equation

$$0 = \max_b \left\{ J_t + [r + b_t(\hat{\mu}_t - r)]X_t J_x + [\lambda - (\lambda + \nu)\pi_t]J_\pi + \frac{1}{2} b_t^2 \sigma^2 J_{xx} + \frac{1}{2} \pi_t^2 (1 - \pi_t)^2 \frac{(\mu_H - \mu_L)^2}{\sigma^2} J_{\pi\pi} + b_t \pi_t (1 - \pi_t) (\mu_H - \mu_L) J_{\pi x} \right\} \quad (9)$$

Thus, the optimal proportion of risky asset is

$$b_t = \frac{\hat{\mu}_t - r}{\sigma^2} \left(\frac{-J_x}{J_{xx} X_t} \right) + \frac{\pi_t(1 - \pi_t)(\mu_H - \mu_L)}{\sigma^2} \left(\frac{-J_{x\pi}}{J_{xx} X_t} \right) \quad (10)$$

Under the power utility function, the optimal portfolio weight is

$$b_t = \frac{(\hat{\mu}_t - r)}{\gamma \sigma^2} + \frac{\pi_t(1 - \pi_t)(\mu_H - \mu_L) g_\pi(\pi_t, t)}{\sigma^2 g(\pi_t, t)} \quad (11)$$

Substitute b_t into the HJB equation, then $g(\pi, t)$ satisfies the following partial differential equation

$$0 = \frac{1 - \gamma}{\gamma} [r + \frac{1}{2\gamma} \frac{(\hat{\mu}_t - r)^2}{\sigma^2}] g + [\lambda - (\lambda + \nu)\pi_t] \frac{1 - \gamma}{\gamma} \pi_t(1 - \pi_t)(\mu_H - \mu_L) \frac{\hat{\mu}_t - r}{\sigma^2} g_\pi + g_\pi + \frac{\pi_t^2(1 - \pi_t)^2(\mu_H - \mu_L)^2}{2\sigma^2} g_{\pi\pi} \quad (12)$$

With $g(\pi, T) = 1, \pi \in [0, 1]$

The solution of Eq. (13) is given by the Feynman-Kac formula

$$g(x, t) = E \left[\exp \left(\frac{1 - \gamma}{\gamma} r(T - t) + \int_t^T \frac{1 - \gamma}{2\gamma} \frac{(\hat{\mu}(s) - r)^2}{\sigma^2} ds \right) \middle| \pi_t = x \right] \quad (13)$$

Where $\hat{\mu}(\pi_t) = \pi_t \mu_H + (1 - \pi_t) \mu_L$.

Given that π_t^x is the solution of Eq. (2) under the initial condition $\pi_0 = x$.

According to Protter [13], π_t^x is differentiable to x . Assuming

that $B_t = \frac{\partial \pi_t^x}{\partial x}$, $g_x(x, t)$ can be expressed as follow based on Eq. (15)

$$g_x(x, t) = E \left[\exp \left(\frac{1 - \gamma}{\gamma} r(T - t) + \int_t^T \frac{1 - \gamma}{2\gamma^2} \frac{(\hat{\mu}(\pi_s) - r)^2}{\sigma^2} ds \right) B_s \middle| \pi_t = x \right] \quad (14)$$

B_t satisfies the following stochastic partial differential equation with $B_0 = 1$.

$$dB_t = -(\lambda + \nu)B_t dt + (1 - 2\pi_t) \frac{\mu_H - \mu_L}{\sigma} B_t dW_t \quad (15)$$

The optimal portfolio in (12) consists of two parts. The first term is the mean-variance portfolio, also known as myopic demand. The second component is the hedging portfolio against the future changes in the state dynamics of the mean return estimates. The hedging portfolio depends on σ and $\mu_H - \mu_L$, $\pi_t(1 - \pi_t)$, $g_\pi(\pi, t) / g(\pi, t)$. The hedging portfolio is large if $\mu_H - \mu_L$ is large and σ is small. Furthermore, if π is near its boundaries and estimation risks are small, the hedging portfolio is small.

The size of the hedging portfolio is the product of $g_\pi(\pi, t) / g(\pi, t)$ and $\pi_t(1 - \pi_t)(\mu_H - \mu_L) / \sigma^2$. Because of the boundary condition $g(\pi, T) = 1$ for all π , for a short-horizon investor, g_π is almost zero for all π , which makes the hedging portfolio is small. For long-horizon investors, $g_\pi(\pi, t) / g(\pi, t)$ can achieve a notable size.

4. Numerical analysis

We use Monte Carlo simulation to estimate g and g_π . They are discretized as follows, Where Z_s^n and B_s^n are n-th sample path of Z_s and B_s respectively.

$$g(\pi, t) = \frac{1}{N} \sum_{n=1}^N \exp\left(\frac{1-\gamma}{\gamma} r(T-t) + \int_t^T \frac{1-\gamma(\hat{\mu}(\pi_s^n) - r)^2}{2\gamma^2 \sigma^2} ds\right) \quad (16)$$

$$g_\pi(\pi, t) = \frac{1}{N} \sum_{n=1}^N \exp\left(\frac{1-\gamma}{\gamma} r(T-t) + \int_t^T \frac{1-\gamma(\hat{\mu}(\pi_s^n) - r)^2}{2\gamma^2 \sigma^2} ds\right) \int_t^T \frac{1-\gamma(\hat{\mu}(\pi_s^n) - r)(\mu_H - \mu_L)}{\gamma^2 \sigma^2} B_s^n ds \quad (17)$$

In table 1, we show optimal portfolio choice under the different state probabilities and risk aversion coefficient for the parameters: $r=0.05$, $\mu_1=0.2$, $\mu_2=0.07$, $\lambda=\nu=0.25$, $\sigma=0.25$, $N=50000$. From part A, when $\gamma>1$, and π_0 increases gradually, investor will invest more wealth in the risky asset. When $\gamma<1$, the optimal portfolio weight increases and then decreases but re-increases with the rise of π_0 . When $\pi_0=0.5$, the optimal portfolio weight decreases to the minimum.

From panel B, at the time of π_0 closer to the boundary, the hedging portfolio decreases. The hedging portfolio position at the time of $\gamma=0.5$ is contrary to that at the time of $\gamma>1$. When $\gamma=0.5$ and $\pi_0>0.6$, the hedging portfolio is positive, and investor invests risk asset; when $\gamma=0.5$ and $\pi_0<0.6$, the hedging portfolio is negative, and investor short sells risk asset. It is the other way around when $\gamma>1$.

Table 1 Optimal portfolio under various risk aversion coefficients

π_0	$\gamma=0.5$	$\gamma=1.5$	$\gamma=2$	$\gamma=4$
Panel A: optimal portfolio weight				
0	0.64	0.213	0.16	0.08
0.3	1.147	0.773	0.633	0.358
0.5	0.343	0.982	0.763	0.404
0.8	4.321	1.285	0.949	0.464
1	4.8	1.6	1.2	0.6
Panel B: hedging portfolio weight				
0	0	0	0	0
0.3	-0.915	0.143	0.161	0.122
0.5	-2.447	0.075	0.083	0.064
0.8	0.353	-0.038	-0.042	-0.032
1	0	0	0	0

Table 2 shows the optimal portfolio and hedging portfolio under different investment horizons for the parameters $\gamma=1.5$, $r=0.05$, $\mu_H=0.2$, $\mu_L=0.07$, $\lambda=\nu=0.25$, $\sigma=0.25$. If $\pi_0>0.5$ is considered as that the market is in the bull market and $\pi_0\leq 0.5$ is bear market, From panel A, the longer the investment horizon in the bear market, the optimal portfolio is larger. Because investor believes the market will pick up gradually although

the current market is in the bear market, the longer the investment horizon, the greater the possibility that market picks up, and investor will increase the investment during the bear market. The situation in the bull market is just the opposite.

From panel B, the longer the investment horizon, the greater the hedging portfolio, because the longer the investment horizon, the greater of the uncertainty of investment opportunity set, investor will pay more attention to this change.

Table 2 Optimal portfolio for various horizons

π_0	T=1	T=5	T=10
Panel A: optimal portfolio weight			
0	0.213	0.213	0.213
0.3	0.621	0.664	0.773
0.5	0.896	0.937	0.982
0.8	1.315	1.331	1.285
1	1.6	1.6	1.6
Panel B: hedging portfolio weight			
0	0	0	0
0.3	-0.008	0.035	0.143
Table 2, cont			
0.5	-0.010	0.029	0.075
0.8	-0.007	0.008	-0.038
1	0	0	0

Table 3 shows optimal portfolio choice for different σ and μ_L . The analysis on σ is in the case of $T=10$, $\gamma=1.5$, $r=0.05$, $\mu_H=0.2$, $\mu_L=0.07$ and $\lambda=v=0.25$; analysis on μ_L is in the case of $T=10$, $\gamma=1.5$, $r=0.05$, $\mu_H=0.2$, $\lambda=v=0.25$ and $\sigma=0.25$. As can be seen from the table, the smaller the σ , the more popular the risky assets if μ_H and μ_L are given. Thus, the smaller the σ , the greater the optimal proportion of risky assets. Panel B indicates that the hedging portfolio is more sensitive to π_0 when σ is getting smaller, because investors are more sensitive to the market state when σ decreases. From the table, we can see that the optimal portfolio is positively correlated with μ_L . The smaller the μ_L , the investment proportion decreases and even the position is negative. Panel B indicates that absolute

of the hedging portfolio is greater than those in other two cases when the expected return is negative in the worst case and $\mu_L=-0.07$.

Table 3 Optimal portfolio for various σ and μ_L

Panel A: optimal portfolio weight				
π_0	$\sigma=0.15$	$\sigma=0.25$	$\mu_L=-0.07$	$\mu_L=0.07$
0	0.593	0.213	-1.28	0.213
0.3	2.613	0.773	-0.092	0.773
0.5	2.870	0.982	0.179	0.981
0.8	3.471	1.285	0.457	1.285
1	4.444	1.6	1.6	1.6
Panel B: hedging portfolio weight				
π_0	$\sigma=0.15$	$\sigma=0.25$	$\mu_L=-0.07$	$\mu_L=0.07$
0	0	0	0	0
0.3	0.865	0.143	0.324	0.143
0.5	0.352	0.075	0.019	0.075
0.8	-0.203	-0.038	-0.566	-0.038
1	0	0	0	0

5. Conclusion

In this paper, we analyze optimal portfolio choice extends the analysis in a continuous-time setting, where the unobservable mean return of a risky asset follows a two-state continuous-time Markov chain. By the dynamic programming and Feynman-Kac formula, the optimal portfolio can be characterized by the expectations of stochastic integrals. The Monte Carlo simulations are implemented to compute the optimal portfolio allocations. The results show that, state uncertainty and dynamic learning have a great influence on optimal portfolio choice. The hedging portfolio of a long-time-horizon investor is larger than that of a short-time-horizon investor. The volatility of risk asset is smaller; the slight change of π_0 will cause a larger change of hedging portfolio. The optimal portfolio allocation can decrease or vary non-monotonically in the investment horizon, which casts in doubt the conventional wisdom that young investor should allocate more wealth into equity. In addition, the optimal portfolio is less sensitive to the change in the state variable for a risk-averse investor.

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