Oscillation for a class of coupled van der Pol oscillators with delays

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Abstract: In this paper, a class of coupled van der Pol oscillators with time delays is investigated. By means of Chafee's criterion, some sufficient conditions to guarantee the existence of oscillatory solutions for the model are obtained. Computer simulations are provided to demonstrate the proposed result

1. Introduction

The dynamical behaviour of coupled oscillators with or without time delays have been studied by many researchers [1-19]. For example, Zhang and Gu have investigated the coupled van der Pol oscillators with two delays:

$$\begin{aligned}x_1''(t) + \varepsilon(x_1^2(t) - 1)x_1'(t) + x_1(t) &= \alpha[y_1(t - \tau_2) - x_1(t)], \\y_1''(t) + \varepsilon(y_1^2(t) - 1)y_1'(t) + y_1(t) &= \alpha[x_1(t - \tau_1) - y_1(t)].\end{aligned}\quad (1)$$

By using the theory of normal form and the center manifold theorem, the explicit expression for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solution were derived. Barron *et al.* have considered the stability and corrective behaviour of a ring of coupled van der Pol oscillators [2, 3]:

$$\begin{aligned}x_1'(t) + \alpha(x_1^2(t) - 1)x_1'(t) + x_1(t) &= b_1[x_n(t) - 2x_1(t) + x_2(t)], \\x_2'(t) + \alpha(x_2^2(t) - 1)x_2'(t) + x_2(t) &= b_2[x_1(t) - 2x_2(t) + x_3(t)], \\&\dots\dots\dots \\x_n'(t) + \alpha(x_n^2(t) - 1)x_n'(t) + x_n(t) &= b_n[x_{n-1}(t) - 2x_n(t) + x_1(t)].\end{aligned}\quad (2)$$

Where b_i is the coupling parameter corresponding to the i th oscillator. If θ_i is the angular position of the i th oscillator, then $\theta_i = i\Delta\theta$, where $\Delta\theta = 2\pi/(n-1)$. The stability analysis is based on the transformation of the linearized equation of the ring into a canonical Hill equation. The collective behaviour is numerically studied. Three values of the oscillators constant are assumed. Wang and Chen have discussed a ring of coupled van der Pol oscillators with time delay coupling as follows:

$$\begin{aligned}u_1''(t) + (\alpha - \beta u_1^2(t))u_1'(t) + au_1(t) &= Au_2'(t - \tau), \\u_2''(t) + (\alpha - \beta u_2^2(t))u_2'(t) + au_2(t) &= Au_3'(t - \tau), \\&\dots\dots\dots \\u_n''(t) + (\alpha - \beta u_n^2(t))u_n'(t) + au_n(t) &= Au_1'(t - \tau).\end{aligned}\quad (3)$$

Where A is the coupling strength, and τ is the time delay. By using the method of multiple scales, the amplitude equations are obtained. Chosen A and τ as the bifurcation parameters, the dynamical

behaviour arising from the bifurcation is classified. Motivated by the above models, in this paper we consider the following general coupled van der Pol system with delays

$$\begin{aligned} u_1''(t) + \varepsilon_1(u_1^2(t) - a_1)u_1'(t) + c_1u_1(t) &= b_1[u_n(t - s_n) - 2u_1(t - s_1) + u_2(t - c_2)], \\ u_2''(t) + \varepsilon_2(u_2^2(t) - a_2)u_2'(t) + c_2u_2(t) &= b_2[u_1(t - s_1) - 2u_2(t - s_2) + u_3(t - c_3)], \\ &\dots\dots\dots \\ u_n''(t) + \varepsilon_n(u_n^2(t) - a_n)u_n'(t) + c_nu_n(t) &= b_n[u_{n-1}(t - s_{n-1}) - 2u_n(t - s_n) + u_1(t - c_1)]. \end{aligned} \quad (4)$$

where ε_i, a_i, b_i , and c_i are constants, time delays $s_i \geq 0 (i = 1, 2, \dots, n)$. Our goal is to investigate the dynamic behaviour of n coupled oscillators. Based on the Chafee's criterion: if a time delay system has a unique unstable equilibrium point, and all solutions of the system are bounded, then this system will generate a limit cycle, namely a periodic solution [20]. Some existent results in the literature have been extended.

2. Preliminaries

It is convenient to write equation (4) as an equivalent $2n$ dimensional first order equation:

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= \varepsilon_1 a_1 x_2(t) - \varepsilon_1 x_1^2(t) x_2(t) - c_1 x_1(t) + b_1[x_{2n-1}(t - \tau_{2n-1}) - 2x_1(t - \tau_1) + x_3(t - \tau_3)], \\ &\dots\dots\dots \\ x_{2n}'(t) &= \varepsilon_n a_n x_{2n}(t) - \varepsilon_n x_{2n-1}^2(t) x_{2n}(t) - c_n x_{2n-1}(t) + b_n[x_{2n-3}(t - \tau_{2n-1}) - 2x_{2n-1}(t - \tau_1) + x_1(t - \tau_1)]. \end{aligned} \quad (5)$$

where time delays $s_i = \tau_{2i-1} (i = 1, 2, \dots, n)$. Obviously, the origin $x_i = 0 (i = 1, 2, \dots, 2n)$ is an equilibrium point of equation (5). The linearization of the coupled equation (5) at origin is

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= \varepsilon_1 a_1 x_2(t) - c_1 x_1(t) + b_1[x_{2n-1}(t - \tau_{2n-1}) - 2x_1(t - \tau_1) + x_3(t - \tau_3)], \\ &\dots\dots\dots \\ x_{2n-1}'(t) &= x_{2n}(t), \\ x_{2n}'(t) &= \varepsilon_n a_n x_{2n}(t) - c_n x_{2n-1}(t) + b_n[x_{2n-3}(t - \tau_{2n-1}) - 2x_{2n-1}(t - \tau_1) + x_1(t - \tau_1)]. \end{aligned} \quad (6)$$

The equations (5) and (6) can be expressed in the following matrix form (7) and (8) respectively:

$$X'(t) = PX(t) + QX(t - \tau) + F(X(t)) \quad (7)$$

$$X'(t) = PX(t) + QX(t - \tau) \quad (8)$$

where $X(t) = [x_1(t), x_2(t), \dots, x_{2n}(t)]^T$, $X(t - \tau) = [x_1(t - \tau_1), 0, x_3(t - \tau_3), 0, \dots, x_{2n-1}(t - \tau_{2n-1}), 0]^T$.

Both $P = (p_{ij})_{2n \times 2n}$ and $Q = (q_{ij})_{2n \times 2n}$ are $2n \times 2n$ matrices.

Lemma 1 Assume that the matrix $R(P + Q)$ is a nonsingular matrix, then equation (6) has a unique equilibrium point, implying that equation (5) has a unique equilibrium point.

Proof An equilibrium point $x^* = [x_1^*, x_2^*, \dots, x_{2n}^*]^T$ of equation (6) is a constant solution of the following algebraic equation:

$$\begin{aligned}
 x_2^* &= 0, \\
 \varepsilon_1 a_1 x_2^* - c_1 x_1^* + b_1 [x_{2n-1}^* - 2x_1^* + x_3^*] &= 0, \\
 &\dots\dots\dots \\
 \varepsilon_n a_n x_{2n}^* - c_n x_{2n-1}^* + b_n [x_{2n-3}^* - 2x_{2n-1}^* + x_1^*] &= 0.
 \end{aligned} \tag{9}$$

The matrix form of (9) is the following equation:

$$Px^* + Qx^* = (P + Q)x^* = Rx^* = 0 \tag{10}$$

Noting that R is a $2n \times 2n$ matrix. According to the linear algebraic knowledge, if R is a nonsingular matrix, equation (9) has only one solution, namely, the trivial solution, implying that equation (6) has a unique equilibrium point. For equation (5), the nonlinear terms $\varepsilon_i x_{2i-1}^2(t) x_{2i}(t)$ ($i = 1, 2, \dots, n$) are high order infinitesimal when x_{2i-1} and x_{2i} ($i = 1, 2, \dots, n$) tend to zero. Therefore, equation (6) has a unique equilibrium point suggests that equation (5) has a unique equilibrium point.

Lemma 2 If parameters ε_i ($i = 1, 2, \dots, n$) are positive values, then all solutions of equation (5) are bounded.

Proof To prove the boundedness of the solutions in equation (5), we construct a Lyapunov function $V(t) = 0.5 \sum_{i=1}^{2n} x_i^2(t)$. Calculating the derivative of $V(t)$ through equation (5) one get

$$\begin{aligned}
 V'(t)|_{(5)} &= \sum_{i=1}^{2n} x_i(t) \dot{x}_i(t) = x_1(t)x_2(t) + x_2(t)\{\varepsilon_1 a_1 x_2(t) + \dots + b_1 x_3(t - \tau_3)\} \\
 &\quad + \dots + x_{2n-1}(t)x_{2n}(t) + x_{2n}(t)\{\varepsilon_n a_n x_{2n}(t) + \dots + b_n [x_{2n-3}(t - \tau_{2n-3}) - \dots + x_1(t - \tau_1)]\} \\
 &= -\varepsilon_1 x_1^2(t)x_2^2(t) - \varepsilon_2 x_3^2(t)x_4^2(t) - \dots - \varepsilon_n x_{2n-1}^2(t)x_{2n}^2(t) + (1 - c_1)x_1(t)x_2(t) \\
 &\quad + b_n x_{2n}(t)[x_{2n-3}(t - \tau_{2n-3}) - 2x_{2n-1}(t - \tau_{2n-1}) + x_1(t - \tau_1)]
 \end{aligned} \tag{11}$$

Noting that as $x_{2i-1}x_{2i}$ ($i = 1, 2, \dots, n$) tend to infinity, $x_{2i-1}^2x_{2i}^2$ ($i = 1, 2, \dots, n$) are higher order infinity than $x_{2i-1}x_{2i}$ ($i = 1, 2, \dots, n$). Since $0 < \varepsilon_i$ ($i = 1, 2, \dots, n$), so there exists suitably large $M > 0$ such that $V'(t)|_{(5)} < 0$ as $|x_i| \geq M$ ($i = 1, 2, \dots, 2n$). This means that all solutions of equation (5) are bounded.

3. Main Results

In order to guarantee the boundedness of the solutions, in the following we always select parameter values $\varepsilon_i > 0$ ($i = 1, 2, \dots, n$).

Theorem 1 Assume that linearized equation (6) has a unique equilibrium point for selected parameter values. Let $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ be $2n$ eigenvalues of matrix p . If there exists some $\alpha_k, k \in \{1, 2, \dots, 2n\}$ such that $\alpha_k > 0$ or the real part $\text{Re}(\alpha_k) > 0$ then the unique equilibrium point of equation (5) is unstable, implying that equation (5) generates a limit cycle, namely, an oscillatory solution.

Proof In order to avoid unnecessary complexity, we restrict ourselves to the case of equation (6)

as $\tau_{2i-1} = \tau_*$ ($1, 2, \dots, n$), where $\tau_* = \text{Min}\{\tau_1, \tau_3, \dots, \tau_{2n-1}\}$. From equation (8) we have the form

$$X'(t) = PX(t) + QX(t - \tau_*) \quad (12)$$

Assume that $\beta_1, \beta_2, \dots, \beta_{2n}$ are $2n$ eigenvalues of matrix Q . Since $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ are $2n$ eigenvalues of matrix P , then the characteristic equation corresponding to equation (12) is the following:

$$\prod_{i=1}^{2n} (\lambda - \alpha_i - \beta_i e^{-\lambda \tau_*}) = 0 \quad (13)$$

Noting that there are n row elements are zeros in matrix Q . Therefore, there is at least one eigenvalue, say $\beta_k = 0$. We are led to an investigation of the nature of the roots for the equation

$$\lambda - \alpha_k - \beta_k e^{-\lambda \tau_*} = \lambda - \alpha_k = 0 \quad (14)$$

The characteristic equation (14) indicates that $\lambda = \alpha_k > 0$. if $\alpha_k > 0$, or $\lambda = \text{Re}(\alpha_k) > 0$ if $\text{Re}(\alpha_k) > 0$. In other words, there exists a positive or a positive real part eigenvalue of characteristic equation (14). Hence, the trivial solution of equation (12) is unstable. Noting that in a time delay system, as the value of delay increases, the instability of the trivial solution still maintains. So for any values of time delays $\tau_{2i-1} \geq \tau_*$ ($1, 2, \dots, n$), the trivial solution of equation (8) (or equation (6)) is unstable. Recall that in equation (5), the nonlinear terms $\varepsilon_i x_{2i-1}^2(t) x_{2i}(t)$ ($i = 1, 2, \dots, n$) are higher order infinitesimal when $x_{2i-1}(t)$ and $x_{2i}(t)$ ($i = 1, 2, \dots, n$) tend to zero. Therefore, the uniqueness and instability of the equilibrium point of equation (6) implies that equation (5) has a unique equilibrium point which is unstable. This means that equation (5) generates a limit cycle, namely, an oscillatory solution of equation (5) [20].

We adopt the following measure and norm of matrices [21]: $u(P) = \max_{1 \leq i \leq n} \{1 + \varepsilon_i - c_i\}$, and

$$\|Q\| = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |q_{ij}| \right\}.$$

Theorem 2 Assume that linearized equation (6) has a unique equilibrium point for selected parameter values. Suppose that the following condition holds

$$(\|Q\|) e^{\tau_*} \exp(-\tau_* |\mu(P)|) > 1 \quad (15)$$

then the unique equilibrium point of equation (5) is unstable, implying that equation (5) generates a limit cycle, namely, an oscillatory solution.

Proof We still consider equation (12). Let $z(t) = \sum_{i=1}^{2n} |x_i(t)|$, then we have

$$z'(t) \leq \mu(P)z(t) + \|Q\| z(t - \tau_*) \quad (16)$$

Specially, for equation

$$y'(t) = \mu(P)y(t) + \|Q\| y(t - \tau_*) \quad (17)$$

If the unique equilibrium point of equation (17) is stable, then the characteristic equation associated with equation (17) given by

$$\lambda = \mu(P) + \|Q\| \exp(-\lambda \tau_*) \quad (18)$$

will have a real negative root say λ_0 , and we have from (18)

$$\begin{aligned} |\lambda_0| + |\mu(P)| &\geq \|Q\| \exp(-\lambda_0 \tau_*) = \|Q\| \exp(|\lambda_0| \tau_*) \\ &= \|Q\| \exp[|\lambda_0| + |\mu(P)| \tau_*] \exp(-|\mu(P)| \tau_*) \end{aligned} \quad (19)$$

Using formula $\exp(|\lambda_0| + |\mu(P)| \tau_*) \geq e(|\lambda_0| + |\mu(P)| \tau_*)$ one can obtain

$$1 \geq \|Q\| e \tau_* \exp(-\tau_* |\mu(P)|) \quad (20)$$

Inequality (20) contradicts (15). Hence, our claim regarding the instability of the trivial solution of equation (17) is valid. According to the comparison principle of differential equation we have $z(t) \leq y(t)$. Since the trivial solution of equation (17) is unstable, implying that the trivial solution of equation (16) is unstable. By Chafee's limit cycle criterion, equation (5) generates an oscillatory solution.

4. Simulation Results

Example 1 First we consider a four coupled van der Pol equation:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \varepsilon_1 a_1 x_2(t) - \varepsilon_1 x_1^2(t) x_2(t) - c_1 x_1(t) + b_1 [x_7(t - \tau_7) - 2x_1(t - \tau_1) + x_3(t - \tau_3)], \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= \varepsilon_2 a_2 x_4(t) - \varepsilon_2 x_3^2(t) x_4(t) - c_2 x_3(t) + b_2 [x_1(t - \tau_1) - 2x_3(t - \tau_3) + x_5(t - \tau_5)], \\ \# \quad \dot{x}_5(t) &= x_6(t), \\ \dot{x}_6(t) &= \varepsilon_3 a_3 x_6(t) - \varepsilon_3 x_5^2(t) x_6(t) - c_3 x_5(t) + b_3 [x_3(t - \tau_3) - 2x_5(t - \tau_5) + x_7(t - \tau_7)], \\ \dot{x}_7(t) &= x_8(t), \\ \dot{x}_8(t) &= \varepsilon_4 a_4 x_8(t) - \varepsilon_4 x_7^2(t) x_8(t) - c_4 x_7(t) + b_4 [x_5(t - \tau_5) - 2x_7(t - \tau_7) + x_1(t - \tau_1)]. \end{aligned} \quad (21)$$

We fix $\varepsilon_1 = 0.0045, \varepsilon_2 = 0.0075, \varepsilon_3 = 0.005, \varepsilon_4 = 0.0065$, and first select time delays as 3.5, 3.6, 3.8, and 3.2. The other parameter values are: $a_1 = 1.15, a_2 = 1.25, a_3 = 1.24, a_4 = 1.32; b_1 = 0.18, b_2 = 0.22, b_3 = 0.28, b_4 = 0.32; c_1 = 1.56, c_2 = 1.45, c_3 = 1.36, c_4 = 1.25$. Corresponding to equation (21), the eigenvalues of matrix P $0.0025 \pm 1.2490i, 0.0030 \pm 1.2042i, 0.0040 \pm 1.1662i$, and $0.0045 \pm 1.1180i$. Since there are eigenvalues of matrix P have positive real parts, it is easy to check the condition of Theorem 1 are satisfied. Equation (21) generates an oscillatory solution (see Fig.1). In order to see the effect of time delays, we increase the time delays as 6.5, 6.6, 6.8, and 6.2, the dynamical behaviour still maintains. However, the oscillatory amplitude and frequency both are changed (see Fig.2).

Example 2 Then we consider a five coupled van der Pol equation:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= \varepsilon_1 a_1 x_2(t) - \varepsilon_1 x_1^2(t) x_2(t) - c_1 x_1(t) + b_1 [x_9(t - \tau_9) - 2x_1(t - \tau_1) + x_3(t - \tau_3)], \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= \varepsilon_2 a_2 x_4(t) - \varepsilon_2 x_3^2(t) x_4(t) - c_2 x_3(t) + b_2 [x_1(t - \tau_1) - 2x_3(t - \tau_3) + x_5(t - \tau_5)], \\
\dot{x}_5(t) &= x_6(t), \\
\dot{x}_6(t) &= \varepsilon_3 a_3 x_6(t) - \varepsilon_3 x_5^2(t) x_6(t) - c_3 x_5(t) + b_3 [x_3(t - \tau_3) - 2x_5(t - \tau_5) + x_7(t - \tau_7)], \\
\dot{x}_7(t) &= x_8(t), \\
\dot{x}_8(t) &= \varepsilon_4 a_4 x_8(t) - \varepsilon_4 x_7^2(t) x_8(t) - c_4 x_7(t) + b_4 [x_5(t - \tau_5) - 2x_7(t - \tau_7) + x_9(t - \tau_9)], \\
\dot{x}_9(t) &= x_{10}(t), \\
\dot{x}_{10}(t) &= \varepsilon_5 a_5 x_{10}(t) - \varepsilon_5 x_9^2(t) x_{10}(t) - c_5 x_9(t) + b_5 [x_7(t - \tau_7) - 2x_9(t - \tau_9) + x_1(t - \tau_1)].
\end{aligned} \tag{22}$$

We fix $\varepsilon_1 = 0.00055, \varepsilon_2 = 0.00065, \varepsilon_3 = 0.00075, \varepsilon_4 = 0.00045$, and $\varepsilon_5 = 0.00036$. Select time delays as 0.6, 0.8, 0.2 and 0.4. The other parameter values are $a_1 = 1.55, a_2 = 1.65, a_3 = 1.25, a_4 = 1.35, a_5 = 1.45; b_1 = 0.25, b_2 = 0.35, b_3 = 0.45, b_4 = 0.55, b_5 = 0.65; c_1 = 0.62, c_2 = 0.52, c_3 = 0.42, c_4 = 0.22, c_5 = 0.32$.

Based on the selected parameter values, corresponding to equation (22) we have $u(P) = 1.0009$, $\|Q\| = 2.2$, and $\tau_* = 0.2$. Then $(\|Q\|) e \tau_* \exp(-\tau_* |u(P)|) = 4.8474 > 1$. The condition (15) is satisfied. Equation (22) generates an oscillatory solution (see Fig.3).

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Fig.1 Oscillation of the solutions, delays: (3.5, 3.6, 3.8, 3.2).

(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$, dashdotted line: $x_4(t)$.

(b) Solid line: $x_5(t)$, dashed line: $x_6(t)$, dotted line: $x_7(t)$, dashdotted line: $x_8(t)$.

Fig.2 Oscillation of the solutions, delays: (6.5, 6.6, 6.8, 6.2).

(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$, dashdotted line: $x_4(t)$.

(b) Solid line: $x_5(t)$, dashed line: $x_6(t)$, dotted line: $x_7(t)$, dashdotted line: $x_8(t)$.