A universal form of internal displacement field based quadrilateral area coordinate method QACM-II

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Keywords: Quadrilateral area coordinate; Generalized conforming; Internal displacement field

Abstract. Additional displacement fields based on internal parameters were used widely in finite element method to increase the order and complementary of displacement polynomials, thus accuracy and insensitivity to mesh distortion of elements can be improved significantly. To ensure that the element could pass patch test, internal displacement field should be formulated without incompatible energy on element’s sides. In this paper, based on generalized conforming theory, the second quadrilateral area coordinate method QACM-II was used to develop universal form for plane elements. This universal internal displacement field can be used easily for those plane elements formulated with QACM-I or QACM-II.

Introduction

Based on Q4 element, professor Wilson [1] developed Q6 in 1973 by introducing additional displacement field formulated with internal parameters, so the displacement polynomials of Q6 element are quadratic complementary about isoparametric coordinate, and its accuracy and insensitivity to mesh distortion were improved significantly. As the additional displacement fields were only restrained by nodal conforming conditions, so Q6 element could not present exact solutions for patch test under irregular meshes.

Inspired by Q6 element, many 4-node non-conforming plane elements had been developed without any problem in convergence, such as the element QM6 proposed by Taylor et al. [2], QP6 by Wachspress [3], NQ6 by Pian et al. [4], the generalized conforming element GC-Q6 by Long et al. [5], the quasi-conforming element QC6 by Chen et al. [6], the hybrid-stress element P-S by Pian et al. [7], etc. All these elements can pass the strict form of patch test and possess excellent performance.

As most of these elements need additional displacement to improve the accuracy, then based on generalized conforming theory, a universal form of additional internal displacement field was developed by using isoparametric coordinate method [8]. This universal form can be used easily for improving the performance of quadrilateral plane elements.

In order to keep the elements insensitive under mesh distortion, Long et al. [9,10] developed the quadrilateral area coordinate QACM-I, Chen et al. [11] and Long et al. [12] developed the quadrilateral area coordinate QACM-II and QACM-III respectively.

In this paper, a universal form of internal displacement field was formulated by using QACM-II based on generalized conforming theory. This universal form can be used easily for those elements formulated with QACM-I or QACM-II, and it would improve the property without changing the compatibility of source elements.

Introduction of QACM-II

Compared with QACM-I, QACM-II contains only two independent coordinate components (Z₁, Z₂). These two components were defined as:

\[ Z₁ = 4 \frac{A₁}{A}, \quad Z₂ = 4 \frac{A₂}{A} \]  \hspace{1cm} (1)

Where A is the area of element, A₁ and A₂ are areas of \( \Delta PM₂M₄ \) and \( \Delta PM₃M₁ \) as shown in Fig.1.
\[A_1 = S(\Delta PM_2M_4)\]
\[A_2 = S(\Delta PM_3M_1)\]  

\(M_i (i=1,2,3,4)\) are the midpoints of elemental sides. Four shape parameters named \(g_i (i = 1, 2, 3, 4)\) were defined in QACM-I as follows:

\[g_1 = \frac{S(\Delta 124)}{A}\]
\[g_2 = \frac{S(\Delta 123)}{A}\]
\[g_3 = 1 - g_1\]
\[g_4 = 1 - g_2\]

Using these shape parameters, the relationship between QACM-II and Cartesian coordinates \((x, y)\) was defined as follows:

\[Z_1 = \frac{1}{A} [\bar{a}_1 + \bar{b}_1 x + \bar{c}_1 y] + (g_2 - g_1)\]
\[Z_2 = \frac{1}{A} [\bar{a}_2 + \bar{b}_2 x + \bar{c}_2 y] + (g_3 - g_2)\]

Where

\[a_i = x_i y_k - x_k y_i, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j, \quad i, j, k = 1, 2, 3, 4\]
\[\bar{a}_i = a_3 - a_1, \quad \bar{b}_i = b_3 - b_1, \quad \bar{c}_i = c_3 - c_1\]
\[\bar{a}_2 = a_4 - a_2, \quad \bar{b}_2 = b_4 - b_2, \quad \bar{c}_2 = c_4 - c_2\]

And \(x_i\) and \(y_i\) are the coordinates of elemental nodes. The transformations of the derivatives of the first order can be written as:

\[
\begin{align*}
\frac{\partial}{\partial x} &= \frac{1}{A} \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial Z_1} \\ \frac{\partial}{\partial Z_2} \end{bmatrix} \\
\frac{\partial}{\partial y} &= \frac{1}{A} \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial Z_1} \\ \frac{\partial}{\partial Z_2} \end{bmatrix}
\end{align*}
\]

(6)

The area integral formulae for the first, second terms are given by:

\[\int_A \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} dA = \frac{A}{3} \begin{bmatrix} g_2 - g_1 \\ g_3 - g_2 \end{bmatrix}\]

(7)
Universal internal displacement field based on QACM-II

In order to pass the patch test, the incompatible energy at element sides induced by internal displacement filed should be zero, then the conforming conditions that internal displacement field should be satisfied are as follows:

\[
\iint \frac{\partial \bar{u}}{\partial x} dA = \iint l \bar{u} ds = 0
\]

\[
\iint \frac{\partial \bar{u}}{\partial y} dA = \iint m \bar{u} ds = 0
\]

\[
\iint \bar{u} dA = 0
\]

where \(l\) and \(m\) are the direction cosine of element sides.

As QACM-II has only two independent components \(Z_1\) and \(Z_2\), the universal displacement field based on internal parameters can be assumed as:

\[
\bar{u} = Z_1^m + \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_n
\]

By substituting Eq.10 into the former two equations of Eq.9, we have:

\[
\iint \frac{\partial \bar{u}}{\partial x} dA = \iint \left( \frac{\bar{b}_1}{A} \frac{\partial \bar{u}}{\partial Z_1} + \frac{\bar{b}_2}{A} \frac{\partial \bar{u}}{\partial Z_2} \right) dA = \frac{n \bar{b}_1}{A} \iint Z_1^{n-1} Z_2^n dA + \frac{m \bar{b}_2}{A} \iint Z_1^n Z_2^{m-1} dA + \bar{b}_1 \alpha_1 + \bar{b}_2 \alpha_2 = 0
\]

and :

\[
\iint \frac{\partial \bar{u}}{\partial y} dA = \frac{n \bar{c}_1}{A} \iint Z_1^{n-1} Z_2^n dA + \frac{m \bar{c}_2}{A} \iint Z_1^n Z_2^{m-1} dA + \bar{c}_1 \alpha_1 + \bar{c}_2 \alpha_2 = 0
\]

With Eq.11 and Eq.12, the undetermined parameters \(\alpha_1\) and \(\alpha_2\) can be derived as :

\[
\begin{align*}
\alpha_1 &= -\frac{1}{A} \left[ \frac{\bar{b}_1}{\bar{c}_1} \right]^{-1} \left[ \frac{\bar{b}_n}{\bar{c}_n} \right] \left\{ \iint Z_1^{n-1} Z_2^n dA \right\} = -\frac{1}{A} \left\{ n \iint Z_1^{n-1} Z_2^n dA \right\} \\
\alpha_2 &= -\frac{1}{A} \left[ \frac{\bar{b}_2}{\bar{c}_2} \right]^{-1} \left[ \frac{\bar{b}_m}{\bar{c}_m} \right] \left\{ \iint Z_1^n Z_2^{m-1} dA \right\} = -\frac{1}{A} \left\{ m \iint Z_1^n Z_2^{m-1} dA \right\}
\end{align*}
\]

By substituting Eq.10 into the third equation of Eq.9, we have:

\[
\iint \bar{u} dA = \iint Z_1^n Z_2^n dA - \frac{n}{A} \iint Z_1^{n-1} Z_2^n dA \iint Z_2 dA - \frac{m}{A} \iint Z_1^n Z_2^{m-1} dA \iint Z_2 dA + \alpha_0 A = 0
\]

Then the undetermined parameters \(\alpha_0\) can be written as:

\[
\alpha_0 = -\frac{1}{A} \iint Z_1^n Z_2^n dA + \frac{n}{A^2} \iint Z_1^{n-1} Z_2^n dA \iint Z_2 dA + \frac{m}{A^2} \iint Z_1^n Z_2^{m-1} dA \iint Z_2 dA
\]

The integration in Eq.15 can be denoted as:

\[
I_{n,m}^Z = \iint Z_1^n Z_2^n dA
\]

Then \(\alpha_0\) and the additional displacement field can be written as follows:

\[
\alpha_0 = -\frac{1}{A} I_{n,m}^Z + \frac{n}{A} I_{n-1,m}^Z I_{1,0}^Z + \frac{m}{A^2} I_{n,m-1}^Z I_{0,1}^Z
\]

\[
\bar{u} = Z_1^n Z_2^n - \frac{n}{A} I_{n-1,m}^Z Z_1 - \frac{m}{A} I_{n,m-1}^Z Z_2 + \alpha_0
\]
Examples of quadratic term are presented as follows:

When $n = 2, m = 0$:

$$
\bar{u} = Z_1^2 + \alpha_1 Z_1 + \alpha_0
$$

$$
\alpha_1 = -\frac{2}{A} \int Z_1 dA = -\frac{2}{3} (g_2 - g_1)
$$

$$
\alpha_0 = -\frac{1}{A} \int Z_1^2 dA + \frac{2}{A^2} \int Z_1 dA \int Z_1 dA = -\frac{1}{3} \left[ (g_3 - g_2)^2 + 1 \right] + \frac{2}{9} (g_2 - g_1)^2
$$

(19)

When $n = 1, m = 1$:

$$
\bar{u} = Z_1 Z_2 + \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_0
$$

$$
\alpha_1 = -\frac{1}{A} \int Z_2 dA = -\frac{1}{3} (g_3 - g_2)
$$

$$
\alpha_2 = -\frac{1}{A} \int Z_1 dA = -\frac{1}{3} (g_3 - g_2)
$$

(20)

$$
\alpha_0 = -\frac{1}{A} \int Z_1 Z_2 dA + \frac{2}{A^2} \int Z_1 dA \int Z_2 dA = -\frac{1}{9} (g_2 - g_1)(g_3 - g_2)
$$

When $n = 0, m = 2$:

$$
\bar{u} = Z_2^2 + \alpha_2 Z_2 + \alpha_0
$$

$$
\alpha_2 = -\frac{2}{A} \int Z_2 dA = -\frac{2}{3} (g_3 - g_2)
$$

(21)

$$
\alpha_0 = -\frac{1}{A} \int Z_2^2 dA + \frac{2}{A^2} \int Z_2 dA \int Z_2 dA = -\frac{1}{3} \left[ (g_2 - g_1)^2 + 1 \right] + \frac{2}{9} (g_3 - g_2)^2
$$

Examples of cubic term are presented as follows:

When $n = 3, m = 0$:

$$
\bar{u} = Z_1^3 + \alpha_1 Z_1 + \alpha_0
$$

$$
\alpha_1 = -\frac{3}{A} \int Z_1^2 dA = -\left[ (g_3 - g_2)^2 + 1 \right]
$$

(22)

$$
\alpha_0 = -\frac{1}{A} \int Z_1^3 dA + \frac{3}{A^2} \int Z_1^2 dA \int Z_1 dA = \frac{2}{15} (g_2 - g_1) \left[ (g_3 - g_2)^2 + 1 \right]
$$

When $n = 2, m = 1$:

$$
\bar{u} = Z_1 Z_2^2 + \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_0
$$

$$
\alpha_1 = -\frac{2}{A} \int Z_1 Z_2 dA = -\frac{2}{3} (g_2 - g_1)(g_3 - g_2)
$$

$$
\alpha_2 = -\frac{1}{A} \int Z_2^2 dA = -\frac{1}{3} \left[ (g_3 - g_2)^2 + 1 \right]
$$

(23)

$$
\alpha_0 = -\frac{1}{A} \int Z_1 Z_2^2 dA + \frac{2}{A^2} \int Z_1 Z_2 dA \int Z_1 dA + \frac{1}{A^2} \int Z_1^2 dA \int Z_2 dA
$$

$$
= \frac{1}{15} \left\{ (g_3 - g_2)^3 + 2 (g_3 - g_2)(g_2 - g_1)^2 + 5 (g_3 - g_2) \right\}
$$

$$
+ \frac{2}{9} (g_3 - g_2)(g_2 - g_1)^2 + \frac{1}{9} (g_3 - g_2) \left[ (g_3 - g_2)^2 + 1 \right]
$$

When $n = 1, m = 2$
\[ \bar{u} = Z_2 Z_2^2 + \alpha_3 Z_1 + \alpha_4 Z_2 + \alpha_0 \]
\[ \alpha_3 = -\frac{1}{A} \int \int Z_2 Z_2^2 dA = -\frac{1}{3} \left( (g_2 - g_1)^2 + 1 \right) \]
\[ \alpha_4 = -\frac{2}{A} \int \int Z_1 Z_2 dA = -\frac{2}{3} (g_2 - g_1)(g_3 - g_2) \]
\[ \alpha_0 = -\frac{1}{A} \int \int Z_1 Z_2^2 dA + \frac{1}{A^2} \int \int Z_2 Z_2 dA \int \int Z_2 dA \int \int Z_2 dA \]
\[ = -\frac{1}{15} \left( (g_2 - g_1)^3 + 2(g_3 - g_2)^2(g_2 - g_1) + 5(g_3 - g_1) \right) \]
\[ + \frac{1}{9} (g_2 - g_1) \left( (g_2 - g_1)^2 + 1 \right) + \frac{2}{9} (g_2 - g_1)(g_3 - g_2)^2 \]

When \( n = 0, m = 3 \)
\[ \bar{u} = Z_2^3 + \alpha_3 Z_2 + \alpha_0 \]
\[ \alpha_3 = -\frac{3}{A} \int \int Z_2^3 dA = -\left[ (g_2 - g_1)^2 + 1 \right] \]
\[ \alpha_0 = -\frac{1}{A} \int \int Z_2^2 dA + \frac{3}{A^2} \int \int Z_2^2 dA \int \int Z_2 dA = \frac{2}{15} (g_3 - g_2) \left[ (g_2 - g_1)^2 + 1 \right] \]

This universal form can be used directly as the shape function of additional displacement based on internal parameters for those plane elements formulated with QACM-I or QACM-II. If the nodal displacement was compatible, the new element would pass the strict form of patch test.

Conclusions

Additional displacement field based on internal parameters are important for developing 4-node plane elements. In this paper, based on generalized conforming conditions, an universal form of internal displacement has been formulated with quadrilateral area coordinate QACM-II, and the quadratic and cubic polynomials are also presented as examples for other researchers. These universal polynomials can be used easily for those plane elements developed with QACM-I or QACM-II.

References