A 4-node plane element based quadrilateral area coordinate method QACM-II

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Keywords: Finite element, Quadrilateral area coordinate, Generalized conforming

Abstract. Nonlinear transformation between the isoparametric and the Cartesian coordinates may make the elements more sensitive to mesh distortion. By using the second kind of quadrilateral coordinate method QACM-II, which can keep the transformation always linear between the area and the Cartesian coordinate systems, a 4-node plane element ACQ\(\lambda\) was developed based on optimized generalized conforming conditions. Additional displacement field based on internal parameters was also used to make the whole displacement field quadratic complementary.

Introduction

As a local coordinate system, isoparametric coordinate method was widely used for developing finite elements as its convenience, such as the most famous elements Q6[1] and QM6[2], but the nonlinear transformation between the isoparametric and the Cartesian coordinates would lead to a problem that the serendipity family was quite sensitive to the mesh distortions.

For overcoming this inherent defect of isoparametric coordinates, Long et al. [3, 4] developed the first kind of quadrilateral area coordinate method QACM-I. For this kind of quadrilateral coordinate method, the transformation between the area and the Cartesian coordinate systems is always linear, so it is powerful for developing new elements which are much more insensitive to mesh distortion.

By using QACM-I, Chen et al. [5] proposed 4-node quadrilateral membrane elements AGQ6-I and AGQ6-II, which are locking free of MacNeal’s thin beam. Based on this research, Cen et al. [6] derived out the analytical element stiffness matrix of AGQ6-I and developed a family of the quadrilateral plane membrane elements [7].

As QACM-I contains four area coordinate components (\(L_1, L_2, L_3, L_4\)), among which only two are independent, Chen et al. [8] proposed the second kind of quadrilateral coordinate method QACM-II which has only two independent coordinate components \(Z_1\) and \(Z_2\).

In this paper, based on generalized conforming theory[9], a 4-node plane element ACQ\(\lambda\) was developed by using QACM-II. Its nodal displacement field was formulated with optimized conforming conditions which would introduce fewest restrains on element’s sides, and additional displacement field based on internal parameters was used to make the whole displacement field quadratic complementary.

Generalized conforming conditions

According to the theory of generalized conforming, the conforming conditions for plane elements under constant stresses are as follows:

 Compatibility of constant stress \(\sigma_x\) :
 \[
\int \varepsilon_x \, d A = \int \frac{\partial u}{\partial x} \, d A = \int \delta u \, ds = \int \delta l \, ds
\]

(1)

 Compatibility of constant stress \(\sigma_y\) :
 \[
\int \varepsilon_y \, d A = \int \frac{\partial v}{\partial y} \, d A = \int \delta v \, ds = \int \delta m \, ds
\]

(2)

 Compatibility of stress \(\tau_{xy}\) :

\[ \int \gamma_y dA = \int \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dA = \int \psi (mu + iv) ds = \int \psi (mu + iv) ds \]

Compatibility of rotation:
\[ \int 2\sigma dA = \int \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA = \int \psi (iv - mu) ds = \int \psi (iv - mu) ds \]

Compatibility of motion along \( x \):
\[ \sum_{i=1}^{4} (u - \partial u^e_i) = 0 \]  (5)

Compatibility of motion along \( y \):
\[ \sum_{i=1}^{4} (v - \partial v^e_i) = 0 \]  (6)

Eq.3 and Eq.4 can be written in equivalent form as follows:
\[ \int \frac{\partial u}{\partial y} dA = \int \psi muds = \int \psi muds \]
\[ \int \frac{\partial v}{\partial x} dA = \int \psi lvds = \int \psi lvds \]

Then there are three conforming conditions for \( u \) and \( v \) respectively, and for a 4-node element, the additional condition was selected as follows:
\[ \sum_{i=1}^{4} (u - \partial u^e_i) \xi_i = 0 \]
\[ \sum_{i=1}^{4} (v - \partial v^e_i) \xi_i = 0 \]

Formulation of nodal displacement

The nodal displacement field of the new element can be assumed with QACM-II as follows:
\[ u_q = \alpha_1 + \alpha_2 Z_1 + \alpha_3 Z_2 + \alpha_4 Z_3 Z_2 \]
\[ v_q = \beta_1 + \beta_2 Z_1 + \beta_3 Z_2 + \beta_4 Z_3 Z_2 \]

With the conforming conditions presented in Eq.1~Eq.8, the undetermined parameters can be formulated out.

Substitute the results of \( \alpha_i (i = 1, 2, 3, 4) \) into Eq.9, then the shape functions of nodal displacement can be written as:
\[ N_{qj} = \frac{1 + 3(g_1 g_3 - g_2 g_4) g_j}{4(1 + g_1 g_3 + g_2 g_4)} \xi_i + \frac{\xi_i}{4} \left[ 1 - \eta_i \left( \frac{g_3 - g_4}{1 + g_1 g_3 + g_2 g_4} \right) \right] Z_j \\
+ \frac{\eta_i}{4} \left[ 1 - \frac{\xi_i}{1 + g_1 g_3 + g_2 g_4} \xi_j \right] Z_j Z_2 \\
+ \frac{3 \xi_i \eta_i}{1 + g_1 g_3 + g_2 g_4} Z_j Z_2 \]

The Partial derivatives of shape functions are as follows:
Then the matrix of strains $B_q$ can be formulated out. The stiffness matrix of nodal displacement field can be written as:

$$K_q = \int_A B_q^T D B_q t dA$$  \hspace{1cm} (12)

Where $D$ is the elastic matrix and $t$ is the thickness of element.

**Formulation of additional displacement**

Additional displacement field is assumed as follows:

$$u_k = \lambda_i N_{\lambda 1} + \lambda_2 N_{\lambda 2}$$

$$v_k = \lambda_i N_{\lambda 1} + \lambda_2 N_{\lambda 2}$$  \hspace{1cm} (13)

The shape functions of internal parameters can be selected as follows:

$$N_{\lambda 1} = Z_i^2 - \frac{2}{3} (g_2 - g_1) Z_i - \frac{1}{3} [(g_3 - g_2)^2 + 1] + \frac{2}{9} (g_2 - g_1)^2$$

$$N_{\lambda 2} = Z_i^2 - \frac{2}{3} (g_3 - g_2) Z_i - \frac{1}{3} [(g_2 - g_1)^2 + 1] + \frac{2}{9} (g_3 - g_2)^2$$  \hspace{1cm} (14)

These two shape functions both can satisfy the conforming condition as:

$$\int_A \frac{\partial N_{\lambda i}}{\partial x} dA = \hat{j} I N_{\lambda i} ds = 0$$

$$\int_A \frac{\partial N_{\lambda i}}{\partial y} dA = \hat{j} m N_{\lambda i} ds = 0$$

$$\int_A N_{\lambda i} dA = 0$$  \hspace{1cm} (15)

With static condensation method, the stiffness matrix of element can be written as:

$$K^e = K_q - K_{\lambda q}^T K_{\lambda q}^{-1} K_{\lambda q}$$

$$K_{\lambda q} = \int_A B_{\lambda q}^T D B_{\lambda q} t dA$$

$$K_{\lambda q} = \int_A B_{\lambda q}^T D B_{\lambda q} t dA$$  \hspace{1cm} (16)

where $B_{\lambda q}$ is the strain matrix about $N_{\lambda i}$, it can be formulated with partial derivatives as follows:

$$\frac{\partial N_{\lambda 1}}{\partial x} = \frac{2}{A} Z_i^2 - \frac{2}{3 A} (g_2 - g_1)$$

$$\frac{\partial N_{\lambda 2}}{\partial x} = \frac{2}{A} Z_i^2 - \frac{2}{3 A} (g_3 - g_2)$$

$$\frac{\partial N_{\lambda 1}}{\partial y} = \frac{2}{A} Z_i^2 - \frac{2}{3 A} (g_2 - g_1)$$

$$\frac{\partial N_{\lambda 2}}{\partial y} = \frac{2}{A} Z_i^2 - \frac{2}{3 A} (g_3 - g_2)$$  \hspace{1cm} (17)

This element model is denoted as ACQ$\lambda 2$. 

\hspace{1cm} (22)
Numerical examples

Example 1. Patch test
The constant strain/stress patch test is shown in Fig. 1. Let Young’s modulus $E=1000$, Poisson’s ratio $\mu=0.25$, and thickness of the patch $t=1$. ACQ$\lambda^2$ can present exact solutions as listed in Tab. 1.

<table>
<thead>
<tr>
<th>Node</th>
<th>$u_i$</th>
<th>$v_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.0</td>
<td>-0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>-0.5</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>-0.125</td>
</tr>
<tr>
<td>5</td>
<td>2.5</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>4.0</td>
<td>0.0</td>
</tr>
<tr>
<td>7</td>
<td>4.0</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Example 2. Cook’s skew beam
A skew cantilever beam under shear distributed load at the free edge, as shown in Fig. 2, was proposed by Cook et al. [10]. The results of vertical deflection at point C, the maximum principal stress at point A and the minimum principal stress at point B are listed in Table 2.

<table>
<thead>
<tr>
<th>Element</th>
<th>$V_C$</th>
<th>$\sigma_{\text{Max}}$</th>
<th>$\sigma_{\text{Min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>11.80</td>
<td>0.1217</td>
<td>-0.0960</td>
</tr>
<tr>
<td>QM6</td>
<td>21.05</td>
<td>0.1928</td>
<td>-0.1580</td>
</tr>
<tr>
<td>HL$^{[1]}$</td>
<td>18.17</td>
<td>0.1582</td>
<td>-0.1335</td>
</tr>
<tr>
<td>P-$S^{[1]}$</td>
<td>21.13</td>
<td>0.1854</td>
<td>-0.1452</td>
</tr>
<tr>
<td>ACQ$\lambda^2$</td>
<td>20.74</td>
<td>0.1936</td>
<td>-0.1452</td>
</tr>
</tbody>
</table>

Example 3. Cantilever beam divided by five quadrilateral elements
A cantilever beam divided by five irregular quadrilateral elements is shown in Fig. 3. Two loading cases are considered: (a) pure bending under moment $M=2000$; (b) shear bending under transverse force $P=150$. The Young’s modulus $E=1500$ and Poisson’s ratio $\mu=0.25$. The results of the vertical deflection $v_A$ at point A and the stress $\sigma_{\text{Max}}$ at point B are given in Tab. 3.
Conclusions

By using the second kind of quadrilateral coordinate method QACM-II, a 4-node plane element ACQλ2 was developed based on generalized conforming theory. Optimized conforming conditions were selected to make the nodal displacement field keep lowest order and fewest restraints were applied on element’s sides. Additional displacement field based on internal parameters was also used to make the whole displacement field quadratic complementary.

References