Bayesian Estimation of Measurement Error Models with Longitudinal Data

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Abstract—In this paper, Bayesian inferences for semiparametric measurement error models (MEs) for longitudinal data are investigated. A semiparametric Bayesian approach combining the stick-breaking prior and the Gibbs sampler together with the Metropolis-Hastings algorithm is developed for simulating observations from the posterior distributions and producing the joint Bayesian estimates of unknown parameters and measurement error. We obtain Bayesian estimations of parameters and covariates subject to MEs. Two simulation studies are presented to illustrate our proposed methodologies.

Keywords—Bayesian; measurement error; longitudinal data

I. INTRODUCTION

Longitudinal data frequently occur in medical follow-up studies and analysis of environmental data, and various methods for analysis these data have been proposed. Lin and Carroll [1]. Measurement error data are often encountered in many fields, including engineering, economics, physics, biology, biomedical science and epidemiology. Statistical inference methods for various parametric MEs have been well established over the past several decades. A systematic survey on this research topic was given and many applications of measurement error data were presented by Fuller (1987) [2] and Carroll, Ruppert, Stefanski and Crainiceanu (2006) [3]. Measurements with errors have been used to study in partially linear models by many scholars, such as Wang, Lin, Gutierrez and Carroll (1998) [4]; Liang, Härdel and Carroll (1999) [5]; Ma and Carroll (2006) [6]; Liang, Wang and Carroll (2007) [7]; Pan, Zeng and Lin (2008) [8]. In this paper, a hybrid algorithm is proposed to generate observations required for a Bayesian inference from the posterior distributions of parameters and covariates subject to ME which is combined with the distribution of Beta and Gamma or Beta and Beta by combining the stick-breaking prior and the Gibbs sampler together with the Metropolis-Hastings algorithm.

II. MEASUREMENT ERROR MODELS

For $i = 1, \dots, n$, suppose that y_i is the observed outcome variable, Z_i be a $r \times 1$ vector of the unobserved covariate variables, and U_i be a $p \times 1$ vector of the observed covariate variables for the i th individual. Let

$Z_i = (X_i^T, U_i^T)^T$, we assume that y_i 's are conditionally independent of each other. For longitudinal data, we consider the generalized linear measurement error models of the following structure

$$p(y_i | X_i, \psi) = \exp \left\{ \frac{y_i \tau_i - d(\tau_i)}{\psi} + c(y_i, \psi) \right\} \quad (1)$$

Where $\gamma_i = E(y_i | X_i) = d'(\tau_i)$, $\mu_i = \text{Var}(y_i | X_i) = \zeta d''(\tau_i)$, ψ is a scale parameter, $d(\cdot)$ and $c(\cdot, \cdot)$ are specific differentiable functions, $d'(\alpha_i) = \partial d(\tau_i) / \partial \tau_i$ and $d''(\tau_i) = \partial^2 d(\tau_i) / \partial \tau_i^2$. The conditional mean γ_i is satisfied the equation

$$\lambda_i = g(\gamma_i) = Z_i^T \alpha_z + U_i^T \alpha_v = X_i^T \alpha \quad (2)$$

Where $g(\cdot)$ is a monotonic differentiable link function, $\beta = (\beta_z^T, \beta_v^T)^T$ is a $(r + p) \times 1$ vector of unknown regression coefficients. As in [9], in this paper, we assume that the true covariate Z_i are measured m times for individual i with additive errors, and Y_i, Z_i are errors free. That is, we cannot observe Z_i but we can observe W_{ij} for $j = 1, \dots, m$ with

$$W_{ij} = Z_i + \theta_{ij} \quad (3)$$

Where the MEs θ_{ij} 's are assumed to follow an unknown distribution, and are independent of Z_i . As in Lachos (2010) [10], we suppose the Dirichlet process (DP) mixture model to specify the distribution of θ_{ij} .

To discriminate specification of the covariate ME model, we define a true covariate model for Z_{ki} ($k = 1, \dots, r$) is

$$Z_{ki} = \beta_{k0} + \beta_{kv}^T U_i + \xi_{ki}, \quad \xi_{ki} \sim N(0, \sigma_z^2). \quad (4)$$

Where ρ_{k0} is an intercept, $\rho_{kv} = (\rho_{k1}, \dots, \rho_{kp})^T$ is a $p \times 1$ vector of unknown regression parameters. Denote $Y = \{y_1, \dots, y_n\}$, $Z = \{Z_1, \dots, Z_n\}$, $U = \{U_1, \dots, U_n\}$, $\theta = \{\theta_1, \dots, \theta_n\}$ and $W = \{W_1, \dots, W_n\}$ in which $Z_i = (z_{i1}, \dots, z_{ri})^T$, $\theta_i = (\theta_{i1}, \dots, \theta_{im})$ and $W_i = \{W_{i1}, \dots, W_{im}\}$ for $i = 1, \dots, n$. Denote $\varepsilon_y = \{\alpha, \psi\}$, $\varepsilon_\beta = \{\beta_{10}, \dots, \beta_{r0}, \beta_{1v}, \dots, \beta_{rv}, \sigma_z^2\}$, ε_θ are parameters of equation (3) and $\varepsilon = \{\varepsilon_y, \varepsilon_\beta, \varepsilon_\theta\}$. The joint probability density function for $\{Y, W, \theta, Z\}$ is expressed as

$$P(Y, W, \theta, Z|U, \varepsilon) = \prod_{i=1}^n \{p(y_i|Z_i, U_i; \varepsilon_y) p(W_i|Z_i; \varepsilon_\theta) p(Z_i|U_i; \varepsilon_\beta)\}. \quad (5)$$

We assume the following priors for parameters α , ψ , $\beta_k = (\beta_{k0}, \beta_{kv}^T)^T$ for $k = 1, \dots, r$ and σ_z^2 are given by

$$\begin{aligned} \alpha | \psi, \alpha^0, H_\alpha^0 &\sim N_{r+p}(\alpha^0, \psi^{-1} H_\alpha^0), \\ \psi^{-1} | b_1, b_2 &\sim \Gamma(b_1, b_2), \\ \beta_k | \beta_k^0, H_{\beta k}^0 &\sim N_{p+1}(\beta_k^0, \psi^{-1} H_{\beta k}^0), \\ \sigma_z^2 | d_1, d_2 &\sim \Gamma(d_1, d_2). \end{aligned}$$

Where $b_1, b_2, \alpha^0, H_\alpha^0, \beta_k^0, H_{\beta k}^0, d_1$ and d_2 are hyperparameters whose values are supposed to be given by the prior information. According to the above presented joint probability density function and priors, a Bayesian approach is considered to make statistical inference on parameters in $\varepsilon = \{\varepsilon_y, \varepsilon_\beta, \varepsilon_\theta\}$ by utilizing the Gibbs sampler together with the Metropolis-Hastings algorithm for our established models.

III. SIMULATION AND BAYESIAN ESTIMATIONS

Basing on the above presented, to illustrate the method proposed we carried out the simulation study, let $n = 200$, $m = 5$, we generate each of 50 replicated data sets $\{(y_i, U_i, W_i, Z_i) : i = 1, \dots, n\}$ from a Poisson distribution with the probability density

$$p(y_i | \gamma_i) = \frac{\gamma_i^{y_i} \exp\{-\gamma_i\}}{y_i!}. \quad (6)$$

Where $\lambda_i = \log(\gamma_i) = Z_i^T \alpha_z + U_i^T \alpha_v = X_i^T \alpha$. Let $U_i \sim N(0, 0.25I_3)$ and Z_{1i} and Z_{2i} in Z_i were generated via Equation (4). In this case, ψ relating to Equation (1) is a constant. The true values of $\alpha_z, \alpha_v, \beta_k$ and σ_z^2 were taken to be $\alpha_z = (0.4, -0.3)^T$, $\alpha_v = (0.3, 0.3, 0.4)^T$, $\beta_k = (0.2, 0.2, 0.2, 0.5)^T$ for $k = 1$ and 2 , and $\sigma_z^2 = 1$, respectively. To test the effectiveness of using the TCDP prior to approximate distributions of MEs $\theta_{ij} = (\theta_{ij1}, \theta_{ij2})^T$, we set the following distributional simulation for θ_{ijk} .

Simulation 1: We specify that $\theta_{ijk} = 0.25(\theta_{ij1} - 1) + 0.75(\theta_{ij2} - 0.4)$ be from $\theta_{ij1} \sim \Gamma(1, 1)$ and $\theta_{ij2} \sim \text{Beta}(2, 3)$.

Simulation 2: We suppose that $\theta_{ijk} = 0.25(\theta_{ij1} - 0.5) + 0.75(\theta_{ij2} - 0.25)$ be from $\theta_{ij1} \sim \text{Beta}(4, 4)$ and $\theta_{ij2} \sim \text{Beta}(1, 3)$.

We considered the following three types of priors for α and β_k to investigate sensitivity of Bayesian estimates to prior inputs.

Type A. The hyperparameters corresponding to the priors of α and β_k were taken to be $\alpha^0 = (0.4, -0.3, 0.3, 0.3, 0.4)^T$, $H_\alpha^0 = 0.25I_5$, $\beta_k^0 = (0.2, 0.2, 0.2, 0.5)^T$ and $H_{\beta k}^0 = 0.25I_4$. This can be regarded as a situation with good prior information.

Type B. The hyperparameters corresponding to the priors of α and β_k were taken to be $\alpha^0 = 1.5 \times (0.4, -0.3, 0.3, 0.3, 0.4)^T$, $\beta_k^0 = 1.5 \times (0.2, 0.2, 0.2, 0.5)^T$, $H_\alpha^0 = 0.75I_5$ and $H_{\beta k}^0 = 0.75I_4$. This can be regarded as a situation with inaccurate prior information.

Type C. The hyperparameters corresponding to the priors of α and β_k were taken to be $\alpha^0 = 0 \times (0.4, -0.3, 0.3, 0.3, 0.4)^T$, $\beta_k^0 = 0 \times (0.2, 0.2, 0.2, 0.5)^T$, $H_\alpha^0 = 10I_5$ and $H_{\beta k}^0 = 10I_4$. This can be regarded as a situation with noninformative prior information.

The simulation results are summarized in Table 1 and Table 2, after 5000 burn-in iterations 5000 observations for each of the generated 50 data sets were collected to evaluate Bayesian estimates via Markov chain Monte Carol (MCMC) samples from the full data posterior distribution. Results under assumption together with three types of prior inputs

were presented in Table 1-2, in Table 1-2 where 'Bias' was the absolute difference between the true value and the mean of the estimates based on 50 replications and 'RMS' was the

root mean square between the estimates based on 50 replications and its true value.

TABLE I. PARAMETER ESTIMATES IN THE FIRST SIMULATION

Parameter	True value	Type Bias	A RMS	Type Bias	B RMS	Type Bias	C RMS
β_{10}	0.2	0.0004	0.0729	0.0146	0.0772	0.0065	0.0669
β_{11}	0.2	0.0394	0.1231	0.0074	0.1436	0.0044	0.1558
β_{12}	0.2	0.0063	0.1459	0.0435	0.1456	0.0432	0.1662
β_{13}	0.5	0.0045	0.1097	0.0228	0.1385	0.0078	0.1645
β_{20}	0.2	0.0119	0.0758	0.0140	0.0649	0.0005	0.0635
β_{21}	0.2	0.0113	0.1272	0.0007	0.1164	0.0207	0.1626
β_{22}	0.2	0.0062	0.1222	0.0163	0.1240	0.0418	0.1571
β_{23}	0.5	0.0148	0.1358	0.0481	0.1577	0.0154	0.1681
α_0	0.4	0.0037	0.0558	0.0221	0.0698	0.0101	0.0526
α_1	-0.3	0.0023	0.0695	0.0086	0.0601	0.0130	0.0621
α_2	0.3	0.0247	0.1193	0.0584	0.1380	0.0163	0.1150
α_3	0.3	0.0206	0.1229	0.0184	0.1348	0.0260	0.1383
α_4	0.4	0.0191	0.1361	0.0018	0.1349	0.0233	0.1337
σ_z^2	1	0.0065	0.0707	0.0011	0.0574	0.0038	0.0696

TABLE II. PARAMETER ESTIMATES IN THE SECOND SIMULATION

Parameter	True value	Type Bias	A RMS	Type Bias	B RMS	Type Bias	C RMS
β_{10}	0.2	0.0197	0.0755	0.0137	0.0771	0.0105	0.0683
β_{11}	0.2	0.0050	0.1297	0.0071	0.1599	0.0137	0.1406
β_{12}	0.2	0.0194	0.1002	0.0004	0.1414	0.0210	0.1647
β_{13}	0.5	0.0009	0.1184	0.0206	0.1506	0.0440	0.1444
β_{20}	0.2	0.0053	0.0820	0.0204	0.0725	0.0115	0.0774
β_{21}	0.2	0.0001	0.1219	0.0320	0.1446	0.0152	0.1570
β_{22}	0.2	0.0279	0.1230	0.0113	0.1195	0.0008	0.1044
β_{23}	0.5	0.0073	0.1220	0.0286	0.1347	0.0139	0.1340
α_0	0.4	0.0054	0.0411	0.0121	0.0502	0.0034	0.0544
α_1	-0.3	0.0053	0.0603	0.0131	0.0662	0.0031	0.0752
α_2	0.3	0.0248	0.1141	0.0333	0.1451	0.0214	0.1110
α_3	0.3	0.0106	0.1136	0.0336	0.1233	0.0218	0.1573
α_4	0.4	0.0106	0.1117	0.0203	0.1172	0.0020	0.1281
σ_z^2	1	0.0086	0.0615	0.0032	0.0698	0.0039	0.0668

IV. CONCLUSION

From Tables 1-2 we can see that (i) Bayesian estimates were reasonably accurate regardless of distributional assumptions of θ_{ijk} and prior inputs of unknown parameters because their Bias values were less than 0.10 and their RMS values were less than 0.20; (ii) the Dirichlet prior can be general enough to capture characteristics of various distributional assumptions for MEs; (iii) the results is conducted to illustrate the distribution of θ_{ijk} can be estimated well by our proposed method.

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