

# Reduced-Dimensional MUSIC Algorithm for Joint Angle and Delay Estimation Based on $L_2$ Norm Constraint in Multipath Environment

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**Abstract**—In this paper, an  $L_2$  norm constraint-based reduced-dimensional MUSIC ( $L_2$ CRD-MUSIC) algorithm for joint angle and delay estimation (JADE) with automatic pair matching in multipath environment is proposed. By utilizing the  $L_2$  norm constraint on the steering vector, the quadratic optimization problem in JADE is decomposed into a one-dimensional (1D) search problem. Thus, angles can be first estimated by a 1D search. For each estimated angle, the matching delay is estimated by the root finding technique. The proposed algorithm has lower computational complexity than the 2D-MUSIC method, but achieves comparable performance with the 2D-MUSIC method. Furthermore, the estimated angles and delays are automatically paired together. Simulation results are presented to verify the effectiveness of the proposed algorithm.

**Keywords**—joint angle and delay estimation (JADE),  $L_2$  norm constraint, orthogonal frequency division multiplexing (ofdm), root finding.

## I. INTRODUCTION

Angle and delay estimation is a classical problem with broad applications in many areas such as wireless communications, sonar and radar. This issue has been extensively studied in recent decades and various algorithms have been proposed for joint estimation of these two parameters, as summarized next.

Maximum likelihood (ML)-based methods [1], ESPRIT based methods [2,3] and MUSIC-based methods [4–6] are the common estimation methods of great interest. Particularly, the ML estimator (MLE) is an optimal technique. However, the direct implementation of grid searching is computationally expensive and prohibitive. Although some iterative algorithms have been developed to reduce the complexity, such as the Gauss Newton-based method in [1], they require accurate initial guesses to avoid the local convergence. ESPRIT-based algorithms [2,3] usually have closed-form solutions and low complexity. In [2], the JADE-ESPRIT algorithm is proposed which utilizes the rotational invariance property of the array, but its performance is significantly worse compared with the Cramer-Rao bound (CRB), especially at low signal-to-noise ratio (SNR). MUSIC-based algorithms [4-6] have better performance than ESPRIT-based algorithms and can be used for some kind of irregularly-spaced array. The two-dimensional MUSIC (2D-MUSIC) method, one of the most widely used high-accuracy algorithms, has been

utilized for JADE [4]. However, it has heavy computational load since the requirement of a 2D search. To reduce the computational complexity, the time-space-time MUSIC (TST-MUSIC) in a tree structure is proposed in [5]. The TST-MUSIC algorithm utilizes the technique of temporal filtering, and it requires three one-dimensional (1D) search processes. In [6], the multi-invariance MUSIC (MI-MUSIC) algorithm is proposed, which transforms the 2D search problem into two 1D search problems. However, it suffers performance degradation due to the approximation on the steering vector. Furthermore, the estimated angles and delays by the MI-MUSIC algorithm cannot be automatically paired together.

In this paper, an  $L_2$  norm constraint-based reduced-dimensional MUSIC ( $L_2$ CRD-MUSIC) algorithm for JADE with automatic pair matching is proposed. The proposed algorithm realizes angle and delay estimation only by a single 1D search, and achieves comparable performance with the 2D-MUSIC algorithm. Simulation results show that the proposed algorithm provides a great tradeoff between performance and complexity. Compared with the 2D-MUSIC algorithm, it requires far lower computational load. In contrast to the MI-MUSIC and JADE-ESPRIT algorithms, it has significantly higher accuracy. Moreover, automatic pair matching is realized by using the root finding technique.

## II. SIGNAL MODEL

In the orthogonal frequency division multiplexing (OFDM) system, consider a multipath environment where signals are received by a uniform linear array (ULA) with  $M$  omnidirectional antennas in the far field. We suppose that the channel is fading but stationary during a short time interval. We suppose that the channel bandwidth is  $B$  and there are  $K$  subcarriers. The discrete channel frequency response of the  $k$ th frequency subcarrier at the  $n$ th time interval is given by [1,2]

$$\mathbf{h}^{(n)}(k) = \sum_{i=1}^Q \mathbf{a}(\theta_i) \beta_i(n) e^{-j2\pi(k-1)\Delta_f \tau_i} \in \mathbb{C}^{M \times 1}, \quad (1)$$

where  $Q$  is the number of propagation paths,  $\theta_i$  is the angle of arrival of the  $i$ th path,  $\beta_i(n)$  is the complex path

fading,  $\Delta_f = B/K$ ,  $\tau_i$  is the time delay to be estimated, and  $\mathbf{a}(\theta_i) = [1, e^{-j2\pi\Delta_d \sin \theta_i / \lambda}, \dots, e^{-j2\pi\Delta_d (M-1) \sin \theta_i / \lambda}]^T$ , in which  $\Delta_d$  is the antenna element spacing,  $\lambda$  is the wavelength, and  $(\cdot)^T$  represents transpose. By stacking  $K$  channel frequency response vector together, we obtain

$$\mathbf{y}(n) = \left[ (\mathbf{h}^{(n)}(1))^T, (\mathbf{h}^{(n)}(2))^T, \dots, (\mathbf{h}^{(n)}(K))^T \right]^T \quad (2)$$

$$= (\mathbf{F} \circ \mathbf{A})\mathbf{b}(n) = \mathbf{U}\mathbf{b}(n) \in \mathbb{C}^{MK \times 1}$$

where  $\circ$  represents the Khatri-Rao product,  $\mathbf{F} = [\mathbf{f}(\tau_1), \mathbf{f}(\tau_2), \dots, \mathbf{f}(\tau_Q)]$ ,  $\mathbf{f}(\tau_i) = [1, e^{-j2\pi\Delta_f \tau_i}, \dots, e^{-j2\pi(K-1)\Delta_f \tau_i}]^T$ ,  $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \dots, \mathbf{a}(\theta_Q)]$ ,  $\mathbf{b}(n) = [\beta_1(n), \beta_2(n), \dots, \beta_Q(n)]^T$ . After collecting  $N$  time interval samples, we have

$$\mathbf{Y} = [\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(N)] \quad (3)$$

$$= \mathbf{U}\mathbf{B} + \mathbf{W} \in \mathbb{C}^{MK \times N}$$

where  $\mathbf{B} = [\mathbf{b}(1), \mathbf{b}(2), \dots, \mathbf{b}(N)]$  and  $\mathbf{W}$  is a white complex Gaussian noise matrix whose covariance matrix is  $\sigma^2 \mathbf{I}$ .

### III. PROPOSED ALGORITHM

The proposed algorithm begins with the computation of the covariance matrix of  $\mathbf{Y}$ , which is estimated as  $\hat{\mathbf{R}} = \mathbf{Y}\mathbf{Y}^H / N$ , in which  $(\cdot)^H$  represents conjugate-transpose. By performing eigenvalue decomposition (EVD) operation for  $\hat{\mathbf{R}}$ , we obtain the noise subspace matrix  $\mathbf{U}_n$  composed of the eigenvectors corresponding to the smallest  $MK - Q$  eigenvalues. According to Eq. 5, the 2D-MUSIC spatial spectrum function for JADE is given by [7]

$$F(\tau, \theta) = \frac{1}{[\mathbf{f}(\tau) \otimes \mathbf{a}(\theta)]^H \mathbf{U}_n \mathbf{U}_n^H [\mathbf{f}(\tau) \otimes \mathbf{a}(\theta)]}, \quad (4)$$

where  $\otimes$  denotes the Kronecker product. Eq. 4 is a complicated non-linear function of the parameters of interest. To avoid the performance degradation and achieve automatic pairing, a reduced-dimensional MUSIC algorithm based on  $l_2$  norm constraint is given as follows. We define

$$\begin{aligned} Z(\tau, \theta) &= [\mathbf{f}(\tau) \otimes \mathbf{a}(\theta)]^H \mathbf{U}_n \mathbf{U}_n^H [\mathbf{f}(\tau) \otimes \mathbf{a}(\theta)] \\ &= \mathbf{f}^H(\tau) [\mathbf{I}_K \otimes \mathbf{a}(\theta)]^H \mathbf{U}_n \mathbf{U}_n^H [\mathbf{I}_K \otimes \mathbf{a}(\theta)] \mathbf{f}(\tau) \quad (5) \\ &= \mathbf{f}^H(\tau) \mathbf{Q}(\theta) \mathbf{f}(\tau), \end{aligned}$$

where  $\mathbf{I}_K$  is the  $K \times K$  identity matrix. Since the  $l_2$  norm of  $\mathbf{f}(\tau)$  is a constant, we consider the constraint of  $\|\mathbf{f}(\tau)\|_2^2 = K$ . Compared with the constraint in the MI-MUSIC method, the  $l_2$  norm constraint is tighter for estimation. Therefore, the estimated results by the l2CRD-MUSIC algorithm are closer to the optimum values. With this constraint, Eq. 5 is transformed into an optimization problem as follows

$$\{\hat{\tau}, \hat{\theta}\} = \arg \min_{\tau, \theta} \mathbf{f}^H(\tau) \mathbf{Q}(\theta) \mathbf{f}(\tau), \quad \text{s.t. } \|\mathbf{f}(\tau)\|_2^2 = K. \quad (6)$$

Using the Lagrange multiplier technique, a cost function is constructed by

$$L(\tau, \theta) = \mathbf{f}^H(\tau) \mathbf{Q}(\theta) \mathbf{f}(\tau) - \alpha (\mathbf{f}^H(\tau) \mathbf{f}(\tau) - K), \quad (7)$$

where  $\alpha$  is a constant. Then we take the gradient of  $L(\tau, \theta)$  with respect to  $\mathbf{f}(\tau)$  and set it to zero, which is given by

$$\frac{\partial L(\phi, \theta)}{\partial \mathbf{f}(\tau)} = 2\mathbf{Q}(\theta) \mathbf{f}(\tau) - 2\alpha \mathbf{f}(\tau) = \mathbf{0}. \quad (8)$$

According to Eq. 8, we obtain

$$\mathbf{Q}(\theta) \mathbf{f}(\tau) = \alpha \mathbf{f}(\tau), \quad (9)$$

which means that  $\alpha$  and  $\mathbf{f}(\tau)$  are the eigenvalue and eigenvector of  $\mathbf{Q}(\theta)$ , respectively. By multiplying  $\mathbf{f}^H(\tau)$  on both sides of Eq. 9, we obtain

$$\mathbf{f}^H(\tau) \mathbf{Q}(\theta) \mathbf{f}(\tau) = \alpha \mathbf{f}^H(\tau) \mathbf{f}(\tau) = \alpha K. \quad (10)$$

According to Eq. 10, we find that when  $\alpha$  is the smallest eigenvalue of  $\mathbf{Q}(\theta)$  and  $\mathbf{f}(\tau)$  is the eigenvector corresponding to the smallest eigenvalue,  $Z(\tau, \theta)$  gets the minimum. We define  $\lambda_{\min}\{\mathbf{Q}(\theta)\}$  and  $\mathbf{v}_{\min}\{\mathbf{Q}(\theta)\}$  as the smallest eigenvalue of  $\mathbf{Q}(\theta)$  and the eigenvector corresponding to the smallest eigenvalue of  $\mathbf{Q}(\theta)$ , respectively. Substituting  $\lambda_{\min}\{\mathbf{Q}(\theta)\}$  and  $\mathbf{v}_{\min}\{\mathbf{Q}(\theta)\}$  into Eq. 6, then angles are estimated by

$$\hat{\theta} = \arg \max_{\theta} \frac{1}{\mathbf{v}_{\min}^H\{\mathbf{Q}(\theta)\} \mathbf{Q}(\theta) \mathbf{v}_{\min}\{\mathbf{Q}(\theta)\}} = \arg \max_{\theta} \frac{1}{K \cdot \lambda_{\min}\{\mathbf{Q}(\theta)\}}, \quad (11)$$

which depends only on  $\theta$ . By searching  $\theta \in (-90^\circ, 90^\circ)$  we can obtain  $\hat{\theta}_i$  ( $i = 1, 2, \dots, Q$ ) which correspond to the  $Q$  largest peaks of Eq. 11.

For each estimated  $\hat{\theta}_i$ , an estimation of the steering vector related to  $\tau_i$  is given by

$$\hat{\mathbf{f}}(\tau_i) = \mathbf{v}_{\min}\{Q(\hat{\theta}_i)\}, \quad i = 1, 2, \dots, Q. \quad (12)$$

We define  $\mathbf{g}(\varphi) = [1, \varphi^1, \dots, \varphi^{K-1}]^T$  where  $\varphi$  is a variable and a fitting function  $P_i(\varphi)$  as follows

$$P_i(\varphi) = \|\mathbf{g} - \hat{\mathbf{f}}(\tau_i)\|_2^2, \quad (13)$$

In the absence of noise,  $P_i(\varphi)$  is equal to zero when  $\varphi = e^{-j2\pi\Delta_f\tau_i}$ . In practice, since the existence of noise,  $P_i(\varphi)$  will be the global minimum when  $\varphi = e^{-j2\pi\Delta_f\tau_i}$ . Defining  $\varphi_i = e^{-j2\pi\Delta_f\tau_i}$ , thus the estimation of  $\varphi_i$  can be obtained by

$$\hat{\varphi}_i = \arg \min_{\varphi} P_i(\varphi), \quad i = 1, 2, \dots, Q \quad (14)$$

Instead of the exhaustive search, we use the root finding technique to find the optimal solution of Eq. 14. By utilizing the power model of  $\mathbf{g}(\varphi)$ ,  $P_i(\varphi)$  can be denoted by

$$P_i(\varphi) = (\mathbf{g}^H(\varphi) - \hat{\mathbf{f}}^H(\tau_i))(\mathbf{g}(\varphi) - \hat{\mathbf{f}}(\tau_i)) = 2K - \sum_{k=1}^K (\varphi^{-(k-1)} \hat{f}_k(\tau_i) + \varphi^{k-1} \hat{f}_k^*(\tau_i)), \quad (15)$$

where  $\hat{f}_k(\tau_i)$  is the  $k$ th element of  $\hat{\mathbf{f}}(\tau_i)$  and  $(\cdot)^*$  denotes complex conjugation. There are  $2(K-1)$  roots for the polynomial in Eq. 15. The estimation of  $\varphi_i$  is equivalent to finding the root inside and closest to the unitary circle of the polynomial in Eq. 15. Assume that the obtained root is  $\hat{\varphi}_i$ , then the estimation of delay is given by

$$\hat{\tau}_i = -\frac{\arg\{\hat{\varphi}_i\}}{2\pi\Delta_f}, \quad i = 1, 2, \dots, Q, \quad (16)$$

where  $\arg\{\cdot\}$  signifies the phase of a complex number. Note that  $\hat{\theta}_i$  and  $\hat{\tau}_i$  are automatically paired together.

If two or more paths have close angles with each other, the accuracy of angle estimation by searching  $\theta$  in Eq. 11

will be deteriorated. In practice, Eq. 5 can also be rewritten as

$$Z(\tau, \theta) = \mathbf{a}^H(\theta) [\mathbf{f}(\tau) \otimes \mathbf{I}_M]^H \mathbf{U}_n \mathbf{U}_n^H [\mathbf{f}(\tau) \otimes \mathbf{I}_M] \mathbf{a}(\theta) \quad (17)$$

$$= \mathbf{a}^H(\theta) \mathbf{P}(\tau) \mathbf{a}(\theta).$$

According to Eq. 17, we can first estimate delays by searching and then estimate angles by the technique of root finding.

The total computational complexity of the proposed algorithm is  $O\{NM^2K^2 + M^3K^3 + N_\theta[(MK-Q)(MK^2 + K^2) + K^3] + Q(K^3 + K^2)\}$ , which is far lower than that of the 2D-MUSIC algorithm whose computational complexity is  $O\{NM^2K^2 + M^3K^3 + N_\theta N_\tau[(MK-Q)(MK+1) + MK]\}$ ,

where  $N_\tau$  stands for the total searching points for delay. The proposed algorithm also has lower complexity than the MI-MUSIC algorithm which costs  $O\{NM^2K^2 + M^3K^3 + N_\theta[(MK-Q)(MK^2 + K^2) + K^3] + N_\tau[(MK-Q)(KM^2 + M^2) + M^3]\}$ . In contrast, the JADE-ESPRIT algorithm costs  $O\{NM^2K^2 + M^3K^3 + 2Q^2(M-1)K + 2Q^2(K-1)M + 6Q^3\}$  which is lower than our algorithm.

#### IV. SIMULATION RESULTS

In this section, to evaluate the performance of the proposed algorithm, 2D-MUSIC, MI-MUSIC, and JADE-ESPRIT methods are compared with the proposed  $l_2$ CRD-MUSIC algorithm. Furthermore, the CRB [4] as a benchmark is also compared with the RMSE of these algorithms. Table 1 presents the specific parameter settings.

TABLE I. SIMULATION PARAMETER SETTINGS

Parameter	Value
Number of antennas $M$	7
Antenna element spacing $\Delta_d$	$\lambda / 2$
Carrier frequency $f_c$	2.4 GHz
Channel bandwidth $B$	20 MHz
Number of subcarriers $K$	64
Number of time intervals $N$	20

Fig. 1 depicts the spectrum of the MI-MUSIC and  $l_2$ CRD-MUSIC algorithms when searching angle at SNR = 0dB. For the sake of brevity and without loss of generality, we consider the case of two propagation paths only [8]. The angles and delays are  $[-20^\circ, 12.5^\circ]$  and  $[0.27\mu s, 0.45\mu s]$ . The fading amplitudes are generated from a zero-mean complex Gaussian distribution with covariance matrix  $\mathbf{RB} = \text{diag}\{[1, 0.8]\}$ . As shown in Fig. 1, the spectrum of the  $l_2$ CRD-MUSIC algorithm is much sharper than that of the

MI-MUSIC algorithm, which means that the  $l_2$ CRD-MUSIC algorithm is superior to the MI-MUSIC algorithm in terms of accuracy. The improvement of accuracy is ascribed to the fact that all the elements of the steering vector are used to construct the constraint in the  $l_2$ CRD-MUSIC algorithm, while the MI-MUSIC algorithm only constrains the first element of the steering vector.

Figs. 2 and 3 present the RMSE of the estimated angles and delays versus SNR, respectively. As is analyzed above, in the  $l_2$ CRD-MUSIC algorithm we can first estimate angles and then estimate delays. Similarly, we can first estimate delays and then estimate angles. As shown in Figs. 2 and 3, the two realizations of the  $l_2$ CRD-MUSIC algorithm have almost the same performance in this situation. Moreover, it is indicated that the robustness to noise is almost the same for the  $l_2$ CRD-MUSIC algorithm compared with the 2D-MUSIC algorithm. The proposed algorithm outperforms the MI-MUSIC and JADE-ESPRIT algorithms in both angle and delay estimations, especially at low SNR.

To further illustrate the good performance of the ISML algorithm, the resolution powers of the different estimators are compared in Figs. 4 and 5. We change the distance between two angles  $\Delta\theta = \theta_2 - \theta_1$  from  $1^\circ$  to  $20^\circ$  with well separated delays and SNR = 0dB. From Fig. 4, we can see that when  $\Delta\theta \leq 5^\circ$ , the performance of the MI-MUSIC in angle estimation will rapidly deteriorate. However, the  $l_2$ CRD-MUSIC algorithm can still guarantee accurate results. The reason is that the  $l_2$ CRD-MUSIC algorithm can first obtain accurate delay estimation by searching and then estimate angles based on the estimated delays. It can be seen from Fig. 5 that all the estimators have good performance in delay estimation. Furthermore, the  $l_2$ CRD-MUSIC algorithm and the 2D-MUSIC algorithm have almost the same performance, and the  $l_2$ CRD-MUSIC algorithm can be considered as a fast implementation of the 2D-MUSIC algorithm.

Fig. 6 shows the complexity of four algorithms versus  $N_\theta$  with  $N_\theta = N_\tau$ . It can be seen clearly that the  $l_2$ CRD-MUSIC algorithm has lower complexity than the 2D-MUSIC and MI-MUSIC algorithms. The reason for complexity reduction by the  $l_2$ CRD-MUSIC algorithm is that it only requires a single 1D search process, however the 2D-MUSIC algorithm requires one 2D search process and the MI-MUSIC algorithm requires two 1D search processes.

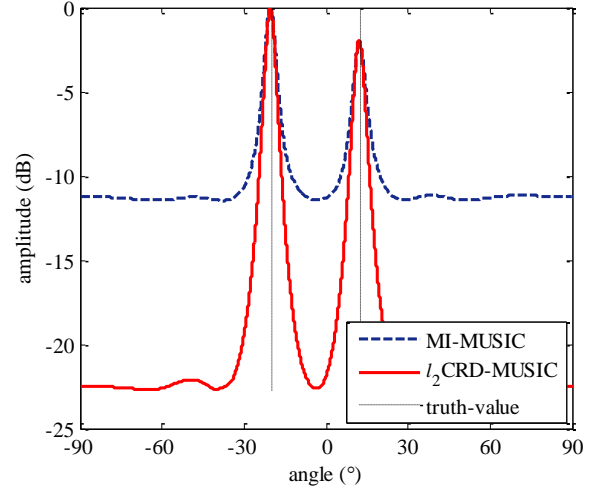


Figure 1. Spectrum for angle search

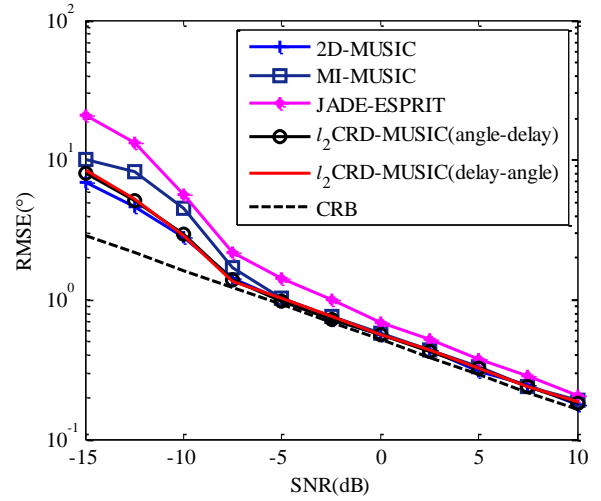


Figure 2. RMSE of angle estimates versus SNR

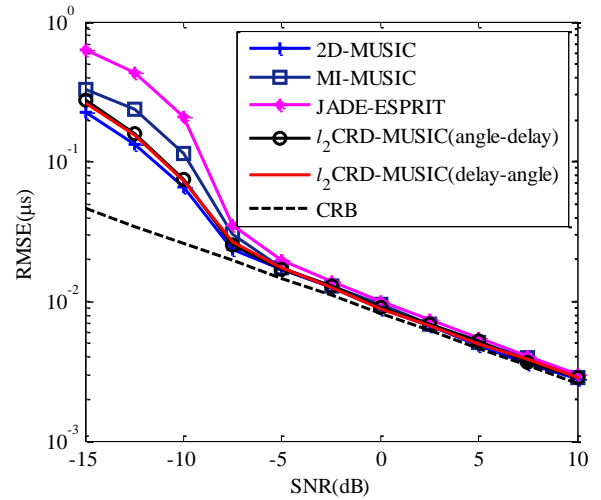


Figure 3. RMSE of delay estimates versus SNR

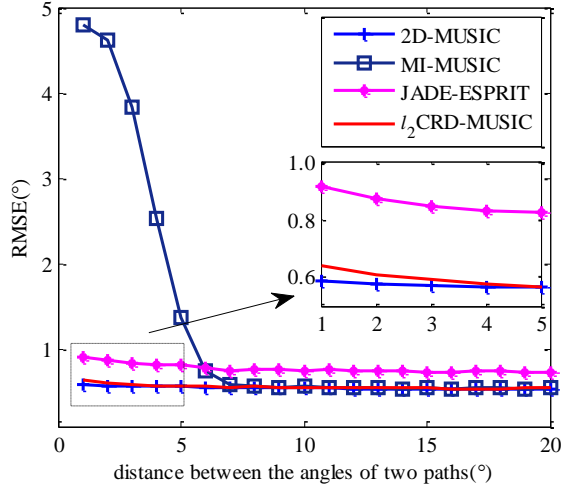


Figure 4. RMSE of angle estimates versus  $\Delta\theta$

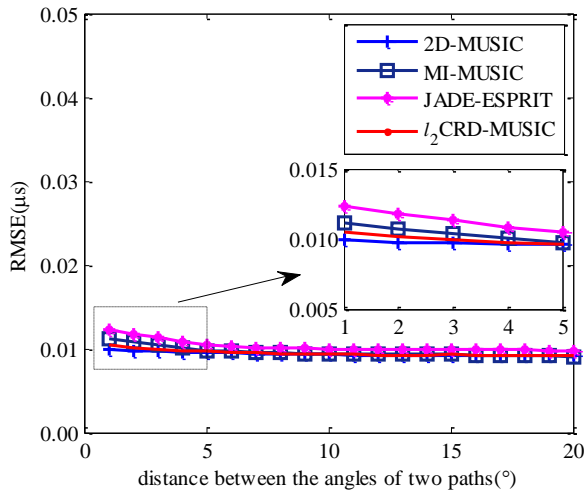


Figure 5. RMSE of delay estimates versus  $\Delta\theta$

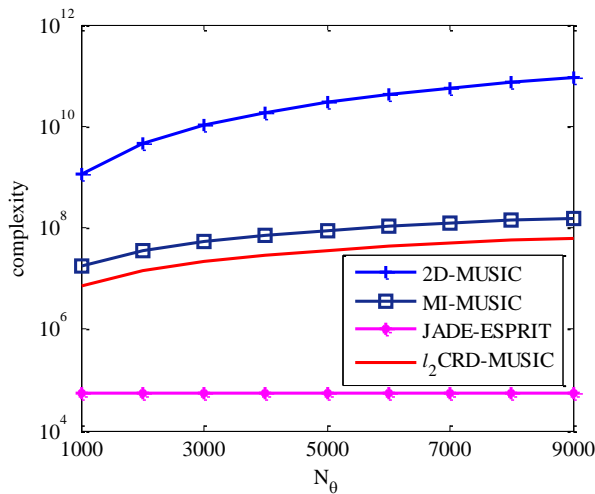


Figure 6. Complexity versus  $N_\theta$  with  $N_\theta = N_r$

## V. CONCLUSIONS

In this paper, we have proposed an  $l_2$ CRD-MUSIC method for JADE. The proposed algorithm, which only requires a single 1D search, has lower complexity than the 2D-MUSIC algorithm, but achieves comparable performance with the 2D-MUSIC algorithm. The proposed algorithm outperforms the MI-MUSIC and JADE-ESPRIT algorithms in terms of accuracy. Furthermore, the pair matching of the estimated angles and delays is automatically achieved. Simulation results demonstrate the effectiveness of the proposed algorithm.

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