

""C'Uqej cule'O qf gnWr f c vpi 'O gj qf 'Y kj 'Wpnpy p" """"""""""""""""""""Rct co gvgf 'F knt kdwkqp

Ma Tianzheng, Lyu hao, Zhang Yimin

Northeastern University, Shenyang

Keywords: Stochastic model updating; Polynomial chaos expansion; Kriging surrogate model; Bayesian method;

Abstract

For stochastic model updating (SMU) problem with unknown parameter probability density function (PDF), an approach is proposed to estimate the statistical properties of the structural parameters. The random variable is represented with polynomial chaos expansion (PCE) and therefore the SMU problem is converted to the evaluation of the PCE coefficients. The Bayesian method is applied to estimate the PCE representation, while the likelihood function in Bayesian calculation is constructed using kernel density estimation (KDE). In order to save computation time, Kriging approach is utilized to establish the surrogate model between structural input and output. A numerical example is given to verify the proposed method.

1 Introduction

The traditional model updating method, which has been widely used in practical engineering, regards the structural parameters as deterministic values. The parameter values are estimated according to experimental results to make the model prediction consistent with measured response. However, uncertainties, which are caused by manufacturing errors, material variability and other factors, widely exist in practical engineering. As the engineering structures become more and more complex, there is an increasing demand for quantifying the influence of uncertainties on structural performance. In recent years, stochastic model updating technique, which utilize the statistical information of experimental observations to calibrate the parameters' probability density function, has drawn researchers' great attention. The methods in references used for SMU include Monte Carlo simulation, perturbation method, Bayesian approach and interval analysis^[1-5].

In this paper, an approach is presented for SMU problem with unknown parameter PDF in practical applications. The structural random variable is decomposed using PCE and the corresponding PCE coefficients are estimated by Bayesian method. Kriging surrogate model between structural input and output is created to save computational cost.

2 Polynomial chaos expansion

Let Z denote second-order random variable (i.e. second moment of Z exists: $E[Z^2] < +\infty$), then Z can be represented by using PCE^[6] as

$$Z = \sum_{i=0}^{P-1} \alpha_i H_i(\xi) \quad (1)$$

Where α_i is coefficient of PCE. P is the highest degree of expansion. $H_i(\cdot)$ is i th-degree Hermite polynomial. ξ denotes standard normal random variable. Eq. (1) can not be directly used in SMU because it's difficult to give an estimation of variation range of the coefficients which will be used in updating.

The mean value μ_Z and standard deviation σ_Z of random variable Z are given by

$$\mu_Z = E[Z] = \alpha_0 \quad (2)$$

$$\sigma_Z^2 = E[(Z - \mu_Z)^2] = \sum_{i=1}^{P-1} \alpha_i^2 \quad (3)$$

Substitute Eqs. (2) and (3) into Eq. (1) yields the following expression

$$Z = \mu_Z + \sigma_Z \sum_{i=1}^{P-1} \gamma_i H_i(\xi) \quad (4)$$

$$-1 \leq \gamma_i = \frac{\alpha_i}{\sigma_Z} \leq 1, i = 1, \dots, P-1$$

Thus, the SMU problem is converted to the estimation of parameter vector $\theta = [\mu_Z, \sigma_Z, \gamma_1, \dots, \gamma_{P-1}]$. It can be observed from Eq. (4) that the variation range of parameter $\gamma_j, j = 1, \dots, P-1$ is $[-1, 1]$. Meanwhile, the variation range of mean value and standard deviation can be obtained based on practical engineering experience easily.

3 Bayesian method

Bayesian method regards the parameters to be estimated as random variables with prior distribution. Parameters' posterior distribution can be evaluated through Bayes formula. Compared with other methods, Bayesian method enable us to make good use of the engineering experience (through prior distribution) and structural observations (through likelihood function) to carry out parameter estimation.

In order to facilitate the following description, let $\{A_i\}_{i=1}^m$ denote $\{A_i, i=1, \dots, m\}$ and let $\{A_{i,j}\}_{i=1, j=1}^{m,n}$ denote $\{A_{i,j}, i=1, \dots, m, j=1, \dots, n\}$.

Let $\omega^m = \{\omega^m\}_{i=1, j=1}^{N_{\text{sample}}, N_{\text{freq}}}$ denote natural frequencies measured from a set of structures manufactured under same conditions. $\omega_{i,j}^m$ is j th order natural frequency of i th structure.

Posterior PDF can be written as

$$\begin{aligned} p(\theta | \omega^m) &\propto p_L(\omega^m | \theta) p_\pi(\theta) \\ p_\pi(\theta) &= \prod_{i=1}^{P+1} p_\pi(\theta_i) \\ p_L(\omega^m | \theta) &= \prod_i \prod_j p_L(\omega_{i,j}^m | \theta) \end{aligned} \quad (5)$$

where $p_\pi(\bullet)$ and $p_L(\bullet)$ are prior distribution and likelihood function, respectively.

Kernel density estimation is employed to calculate the likelihood function. First, generate N_0 samples $\{\xi_k\}_{k=1}^{N_0}$ from standard normal distribution Normal. Then, for prescribed parameter θ , substitute $\{\xi_k\}_{k=1}^{N_0}$ into Eq. (4) to obtain N_0 structural input samples $\{z_k\}_{k=1}^{N_0}$ and compute the corresponding structural frequencies $\{\tilde{\omega}_{k,j}\}_{k=1, j=1}^{N_0, N_{\text{freq}}}$ at each input sample.

Thus, according to KDE principle, the probability density function for j th order frequency $f_j(\omega)$ can be expressed as

$$\begin{aligned} f_j(\omega) &= \frac{1}{N_0 h_j} \sum_{k=1}^{N_0} K\left(\frac{\omega - \tilde{\omega}_{k,j}}{h_j}\right) \\ h_j &= 1.06 \hat{\sigma}_j N_0^{-1/5} \\ \hat{\sigma}_j &= \sqrt{\frac{1}{N_0 - 1} \sum_{k=1}^{N_0} \left[\tilde{\omega}_{k,j} - \frac{1}{N_0} \sum_{k=1}^{N_0} \tilde{\omega}_{k,j} \right]^2} \end{aligned} \quad (6)$$

Substitute Eq. (6) into Eq. (5) leads to the following expression

$$\begin{aligned} p(\theta | \omega^m) &\propto p_\pi(\theta) \prod_i \prod_j f_j(\omega_{i,j}^m) \\ &= p_\pi(\theta) \prod_i \prod_j \left[\frac{1}{N_0 h_j} \sum_{k=1}^{N_0} K\left(\frac{\omega_{i,j}^m - \tilde{\omega}_{k,j}}{h_j}\right) \right] \end{aligned} \quad (7)$$

Markov chain Monte Carlo (MCMC) simulation is applied to perform the evaluation of Eq. (7). Once the posterior PDF is obtained, the parameter θ value can be inferred based on MAP (maximum a posteriori probability estimate) principle.

3 Kriging surrogate model

Conducting MCMC simulation needs to compute the structural natural frequency at different input repeatedly. In order to save computational cost, Kriging approach is used to build a surrogate model.

Kriging model can be written as

$$\begin{aligned} y(\mathbf{x}) &= \mathbf{f}^T(\mathbf{x}) \boldsymbol{\beta} + g(\mathbf{x}) \\ \mathbf{f}(\mathbf{x}) &= [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_p(\mathbf{x})] \\ \boldsymbol{\beta} &= [\beta_1, \beta_2, \dots, \beta_p] \end{aligned}$$

where \mathbf{x} is the system input column vector of dimension d .

$y(\mathbf{x})$ represent the system output. $\mathbf{f}^T(\mathbf{x}) \boldsymbol{\beta}$ is global approximation. $\mathbf{f}(\mathbf{x})$ is polynomial function vector. $\boldsymbol{\beta}$ is coefficient vector to be estimated. $g(\mathbf{x})$ is Gauss process, with mean value 0 and variance σ^2 , to represent random error. The covariance function of $g(\mathbf{x})$ is

$$\begin{aligned} \text{Cov}(g(\mathbf{u}), g(\mathbf{v})) &= \sigma^2 R(\mathbf{u}, \mathbf{v}; \boldsymbol{\theta}) \\ &= \sigma^2 \prod_{j=1}^d \exp(-\rho_j |u_j - v_j|^2) \end{aligned}$$

where u_j and v_j represent i th component of input \mathbf{u} and \mathbf{v} , respectively.

Let $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ denote n training points for Kriging model. For predetermined $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d)$, $\boldsymbol{\beta}$ and σ can be estimated as

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{R}^{-1} \mathbf{Y} \\ \hat{\sigma}^2 &= \frac{1}{N} (\mathbf{Y} - \mathbf{F} \hat{\boldsymbol{\beta}})^T \mathbf{R}^{-1} (\mathbf{Y} - \mathbf{F} \hat{\boldsymbol{\beta}}) \\ \mathbf{F} &= \begin{bmatrix} f_1(\mathbf{x}_1), f_2(\mathbf{x}_1), \dots, f_p(\mathbf{x}_1) \\ \vdots \\ f_1(\mathbf{x}_n), f_2(\mathbf{x}_n), \dots, f_p(\mathbf{x}_n) \end{bmatrix} \\ \mathbf{Y} &= [y(\mathbf{x}_1), y(\mathbf{x}_2), \dots, y(\mathbf{x}_n)]^T \\ \mathbf{R} &= \begin{bmatrix} R(\mathbf{x}_1, \mathbf{x}_1), \dots, R(\mathbf{x}_1, \mathbf{x}_n) \\ \vdots & \ddots & \vdots \\ R(\mathbf{x}_n, \mathbf{x}_1), \dots, R(\mathbf{x}_n, \mathbf{x}_n) \end{bmatrix} \end{aligned}$$

where F is coefficient matrix. R is structural output matrix. R is correlation matrix. Correlation parameter ρ can be evaluated as

$$\hat{\rho} = \arg \max \left\{ -\frac{\ln(\det(R)) + n \ln(\hat{\sigma}^2)}{2} \right\}$$

After Kriging model established, for arbitrary unknown structural input x^* , the structural output y^* predicted by Kriging model can be expressed as

$$y^*(x^*) = f^T(x^*)\hat{\beta} + r^T(x^*)R^{-1}(Y - F\hat{\beta})$$

$$r^T = [R(x^*, x_1), \dots, R(x^*, x_N)]$$

4 Numerical example

The beam with rectangular section is shown in Fig. 1. The elastic modulus is considered as random variables and its corresponding probability distribution information is shown in Table 1. The beam is modeled with FEM software Nastran. Monte Carlo simulation is used to generate 200 samples simulated as measured output for 3 cases in Table 1, respectively. The first four frequencies of the beam are used.

Kriging model is constructed with 30 training points generated with Latin hypercube sampling(LHS). Uniform distribution is selected as prior distribution for Bayesian estimation. MCMC simulates 10000 samples and the first 5000 samples are abandoned as burn-in period. The identified mean value and standard deviation of elastic modulus are shown in Table 2 and correspondingly, the identified PDF is shown in Fig. 2.

It is seen from Table 2 that the identified mean value and standard deviation converge relatively well to real values. However, the identified value of mean is more accurate than standard deviation. Fig. 2 shows that the identified PDFs are able to approximate true PDFs well.

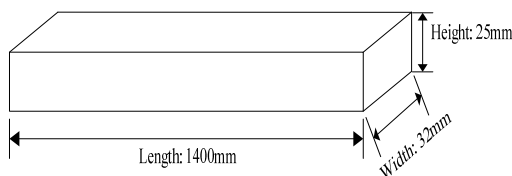


Fig.1 Schematic of beam

	Distribution	Mean	Standard deviation
Case 1	Normal	206.00	6.0000
Case 2	Lognormal	205.76	6.1728
Case 3	Weibull	207.81	6.8854

Table.1 Probability distribution characteristics of elastic modulus

	Distribution	Mean	Standard deviation
Case 1	Normal	206.7240	6.6940
Case 2	Lognormal	205.1880	6.9320
Case 3	Weibull	207.5200	7.8660

Table.2 Results of stochastic model updating

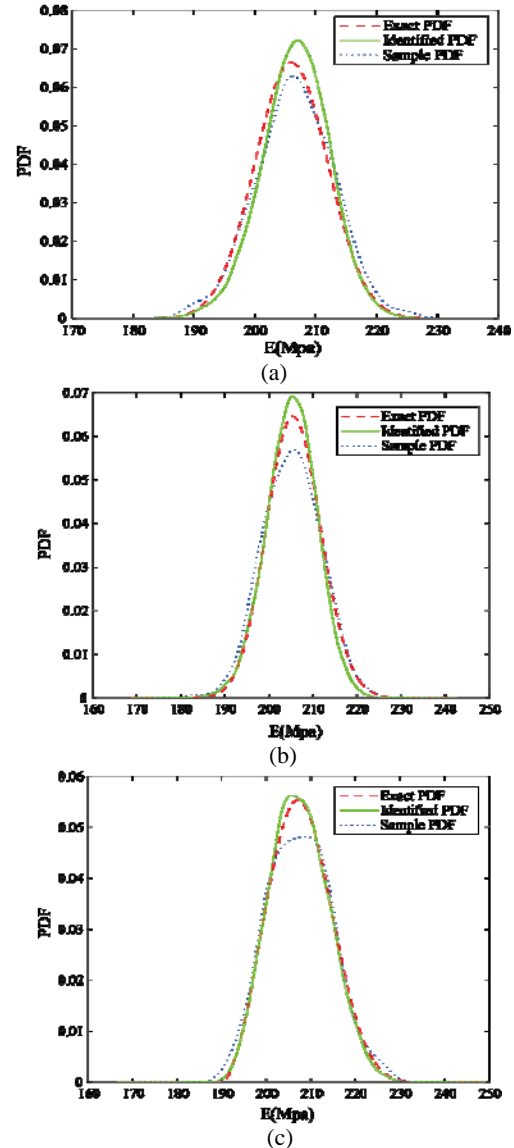


Fig.2 Identified PDF of elastic modulus:(a)Case 1; (b)Case 2; (c)Case 3 .

4 Conclusion

In this paper, an approach for SMU with unknown parameter distribution, which combines polynomial chaos expansion, Bayesian parameter estimation, kernel density estimation and Kriging surrogate model, is presented. A numerical model is given to demonstrate the feasibility of the method. In addition, the proposed approach can be applied to questions with large parameter variation.

Acknowledgements

We would like to express our appreciation to Chinese National Natural Science Foundation (51605083).

References

- [1] Mottershead J E, Mares C, Friswell M I. Stochastic model updating: Part 1-theory and simulated example, *Mechanical Systems and Signal Processing*, 20, pp. 1674-1695, (2006).
- [2] Khodaparast H H, Mottershead J E, Friswell M I. Perturbation methods for the estimation of parameter variability in stochastic model updating, *Mechanical Systems and Signal Processing*, 22, pp. 1751-1773, (2008).
- [3] Govers Y, Link M. Stochastic model updating-Covariance matrix adjustment from uncertain experimental modal data, *Mechanical Systems and Signal Processing*, 24, pp. 696-706, (2010).
- [4] Wan H P, Ren W X. Stochastic model updating utilizing Bayesian approach and Gaussian process model , *Mechanical Systems and Signal Processing*, 70-71, pp. 245-268, (2016).
- [5] Khodaparast H H, Mottershead J E, Badcock K J. Interval model updating with irreducible uncertainty using the Kriging predictor, *Mechanical Systems and Signal Processing*, 25, pp. 1204-1226, (2011).
- [6] Le Maître O P, Knio O M. Spectral methods for uncertainty quantification, New York, Springer, (2010).