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Transmit Beamspace Design For Unitary ESPRIT In Two Dimensional Hybrid MIMO Phased-Array Radar

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Abstract

In this paper, a transmit beamspace design method for two dimensional Hybrid MIMO phased-array radar in application of direction of arrival (DOA) estimation is proposed. A new method to construct the virtual array rotational invariance property that enables the use of search-free DOA estimation techniques at receiver is introduced. Semidefinite relaxation is used to transform the proposed formulation into a convex one that can be solved efficiently. The unitary ESPRIT is also introduced. Numerical simulation validate the proposed approach.

1 Introduction

The concept of multiple-input multiple-output (MIMO) radar has been researched in many aspects [1]. It offers significant performance improvements such as higher sensitivity to detect targets, higher angular resolution, increasing number of detectable targets [2] and combating the signal scintillation [3]. However, MIMO radar sacrifices transmit coherent processing gain as compared to phased-array radar [4]. This can lead to signal-to-noise radio (SNR) loss in the receive array which then influence DOA estimation accuracy when practical application. Focusing on the MIMO radar with collocated arrays which can increase aperture by forming virtual arrays, Hybrid MIMO phased-array radar is proposed to add transmit coherent processing gain in the traditional MIMO radar [5]. The essence of this technique is dividing the transmit array into several subarrays, then one waveform can be transmitted coherently from each element of the same subarray, different orthogonal waveforms are transmitted from different subarrays. Combining the advantages of MIMO radar and phased radar, Hybrid MIMO Phased-array radar can improve the performance of DOA estimation.

Recently, some algorithms have been developed for DOA estimation in the context of Hybrid MIMO Phased-array radar with linear M collocated transmit antennas and N receive antennas in one dimensional. The MUSIC algorithm proposed in [6] require an exhaustive search over unknown parameters and it will result in high computational complexity. The search-free ESPRIT-based algorithm of [7] needs rotational invariance property in the virtual array as a precondition, and it needs additional parameter matching algorithm when there are more than one targets in the space. Based on the research above， some scholars begin to research on the DOA estimation of two dimensional Hybrid MIMO phased-array radar, which has M×N collocated transmit antennas and M×N receive antennas. [8] utilizes MUSIC algorithm in multiple targets estimation, especially, the high computational complexity from the exhaustive search will be more lager in two dimensional array with more antennas. Hence, how to apply the DOA estimation on two dimensional Hybrid MIMO phased-array radar in a more efficient way is a problem which is waiting to be solved.

In this paper, we aim at improving DOA estimation quality of two dimensional Hybrid MIMO phased-array radar both in transmit beampattern design and DOA algorithm improvement. Firstly, we design the transmit weight matrix to force the energy on the desired targets spatial range under the constraint of uniform power distribution. We will also form the rotational invariance of the virtual array. Meanwhile, we wish to control the side lobe and form nulls in the interference direction of the transmit beampattern. Then, we will introduce how to use Unitary ESPRIT (which behaves more excellent than the traditional ESPRIT in many aspects, such as: lower computational complexity, higher estimation accuracy [9]) on the joint azimuth and elevation estimation. finally, corresponding simulation results and analysis are given.

The paper is organized as follow. Two dimensional Hybrid MIMO phased-array radar signal model is briefly introduced In section II, transmit weight matrix design method is given in section III. In section IV, Unitary ESPRIT algorithm is introduced. In section V, simulation examples for the proposed method are drawn.

2 System model

Consider a uniform rectangular array (URA) two dimensional Hybrid MIMO phased-array radar system with $M_t \times N_t$ antennas in the transmit array and $M_r \times N_r$ antennas in the receive array. Where M_t is the number of antenna elements in column, and N_t is the number of antenna elements in row. The elements on any given column in the transmit array are assumed to be equally spaced. Where d_m and d_n are the distances between the adjacent antennas at each column and at each row respectively. Let the steering vector of the each transmit subarray represented as

$$\boldsymbol{a}_k(\theta,\phi)=\mathrm{vec}(\boldsymbol{Z}_k \odot[\boldsymbol{\mu}(\theta,\phi)\boldsymbol{\upsilon}^{\mathrm{T}}(\theta,\phi)]) \tag{1}$$

where $\boldsymbol{Z}_k$ is an matrix of ones and zeros where the mn^{th} entry equals zero if the mn^{th} element of the array is absent, $\mathrm{vec}(\cdot)$ stands for the operator that stacks the columns of a matrix in one column vector, $\odot$ stands for the Hadamard product, $(\bullet)^{\mathrm{T}}$ denotes the transpose, θ and ϕ denote the elevation and azimuth angles, respectively. $\boldsymbol{\mu}(\theta,\phi)$ and $\boldsymbol{\upsilon}(\theta,\phi)$ are defined as follows

$$\boldsymbol{\mu}(\theta,\phi)=[1,\mathrm{e}^{\mathrm{j}2\pi d_m \sin(\theta)\cos(\phi)},\cdots,\mathrm{e}^{\mathrm{j}2\pi(M_t-1)d_m \sin(\theta)\cos(\phi)}]^{\mathrm{T}} \tag{2}$$

$$\boldsymbol{\upsilon}(\theta,\phi)=[1,\mathrm{e}^{\mathrm{j}2\pi d_n \sin(\theta)\sin(\phi)},\cdots,\mathrm{e}^{\mathrm{j}2\pi(N_t-1)d_n \sin(\theta)\sin(\phi)}]^{\mathrm{T}} \tag{3}$$

Let $\boldsymbol{\varphi}(t)=[\varphi_1(t),\cdots,\varphi_K(t)]^{\mathrm{T}}$ be the vector of predesigned independent waveforms which satisfy the orthogonality condition $\int_{T_0}\boldsymbol{\varphi}(t)\boldsymbol{\varphi}(t)^{\mathrm{H}}=\boldsymbol{I}_K$, $\boldsymbol{I}_K$ is the identity matrix of size $K\times K$. The complex envelope of the signals transmitted by k^{th} subarray is

$$\boldsymbol{s}_k(t)=\sqrt{\frac{M_t N_t}{K}}\boldsymbol{w}_k\varphi_k(t) \tag{4}$$

where $\boldsymbol{w}_k$ is the transmit weight vector used to form the k^{th} transmit beam. The 2D transmit beampattern can be written as

$$G_t(\theta,\phi)=\frac{M_t N_t}{K}\sum_{k=1}^{K}\boldsymbol{a}_k^{\mathrm{H}}(\theta,\phi)\boldsymbol{w}_k\boldsymbol{w}_k^{\mathrm{H}}\boldsymbol{a}_k(\theta,\phi) \tag{5}$$

Assume a target at direction of (θ_t,ϕ_t) , the signal reflected by the target is

$$r(t,\theta_t,\phi_t)=\sqrt{M_t N_t/K}\,\beta_t\sum_{k=1}^{K}\boldsymbol{w}_k{}^{\mathrm{H}}\boldsymbol{a}_k(\theta_t,\phi_t)\mathrm{e}^{-\mathrm{j}\tau_k(\theta_t,\phi_t)}\varphi_k(t) \tag{6}$$

where β_t is the reflection coefficient associated with the target. $\tau_k(\theta_t,\phi_t)$ is the time required for the signal to cover from the distance between the first element of the k^{th} subarray. In addition to the desired targets, we also assume ther eare L interfering targets at (θ_i,ϕ_i) .The received data vector at receive array can be modeled as

$$\boldsymbol{x}(t)=r(t,\theta_t,\phi_t)\boldsymbol{b}(\theta_t,\phi_t)+\sum_{i=1}^{L}r(t,\theta_i,\phi_i)\boldsymbol{b}(\theta_i,\phi_i)+\boldsymbol{n}(t) \tag{7}$$

where $\boldsymbol{b}(\theta,\phi)$ is the $M_r N_r\times 1$ receive steering vector and $\boldsymbol{n}(t)$ is the noise that is supposed to have zero mean. By applying matched filtering to the received data vector for each of the orthogonal waveforms $\varphi_k(t)$ $k=1,2\cdots K$ .The $KM_r N_r\times 1$ virtual receive data is

$$\begin{aligned}\boldsymbol{y}&=\int_{T_0}\boldsymbol{x}(t)\boldsymbol{\varphi}_k^{*}(t)dt\\&=\sqrt{\frac{M_t N_t}{K}}\beta_t\boldsymbol{u}(\theta_t,\phi_t)+\sum_{i=1}^{L}\sqrt{\frac{M_t N_t}{K}}\beta_i\boldsymbol{u}(\theta_i,\phi_i)+\hat{n}\end{aligned} \tag{8}$$

where $\hat{n}=\int_{T_0}\boldsymbol{n}(t)\boldsymbol{\varphi}_k^{*}(t)dt$ is the $KM_r N_r\times 1$ noise term whose covariance is $\sigma_n^2\boldsymbol{I}_{KM_r N_r\times 1}$.

3 Transmit beamspace design

In order to use Unitary ESPRIT in DOA estimation, we construct rotational invariance property firstly.

The $KM_r N_r\times 1$ virtual receive data $\boldsymbol{y}$ can be divided into K signal subarrays $\boldsymbol{y_1}\cdots\boldsymbol{y_K}$ through the following

$$\begin{bmatrix}\boldsymbol{y}_1\\ \vdots\\ \boldsymbol{y}_K\end{bmatrix}=\sqrt{\frac{M_t N_t}{K}}\beta_t\begin{bmatrix}\boldsymbol{w}_1^{\mathrm{H}}\boldsymbol{a}_1(\theta,\phi)\mathrm{e}^{-\mathrm{j}\tau_1(\theta,\phi)}\boldsymbol{b}(\theta,\phi)\\ \vdots\\ \boldsymbol{w}_K^{\mathrm{H}}\boldsymbol{a}_K(\theta,\phi)\mathrm{e}^{-\mathrm{j}\tau_K(\theta,\phi)}\boldsymbol{b}(\theta,\phi)\end{bmatrix} \tag{9}$$

any two subarrays $\boldsymbol{y_k}$, $\boldsymbol{y_i}$ satisfy

$$\boldsymbol{y_k}=\mathrm{e}^{\mathrm{j}(\tau_k(\theta,\phi)-\tau_i(\theta,\phi))}\frac{\left|\boldsymbol{w_k}^{\mathrm{H}}\boldsymbol{a}_k(\theta,\phi)\right|}{\left|\boldsymbol{w_i}^{\mathrm{H}}\boldsymbol{a}_i(\theta,\phi)\right|}\boldsymbol{y_i} \tag{10}$$

where $i,k=1,\cdots,K$ .The rotational invariance property can be enforced by imposing the constraint $\left|\boldsymbol{w_k}^{\mathrm{H}}\boldsymbol{a}_k(\theta,\phi)\right|=\left|\boldsymbol{w_i}^{\mathrm{H}}\boldsymbol{a}_i(\theta,\phi)\right|$ while designing the transmit weight matrix. If K=2, The rotational invariance property can be enforced only by forcing $\left|\boldsymbol{w}_1^{\mathrm{H}}\boldsymbol{a}_1(\theta,\phi)\right|=\left|\boldsymbol{w}_2^{\mathrm{H}}\boldsymbol{a}_2(\theta,\phi)\right|$, otherwise , it will be $\left|\boldsymbol{w}_i^{\mathrm{H}}\boldsymbol{a}_i(\theta,\phi)\right|=\left|\boldsymbol{w}_{i+K/2}^{\mathrm{H}}\boldsymbol{a}_{i+K/2}(\theta,\phi)\right|$, where $i=1,2\cdots K/2$.

In addition to design the rotational invariance property in the virtual array, we can also optimize transmit beampattern by transmit weight matrix design. The main idea is to design the transmit weight matrix which aims at minimizing the difference between a desired transmit beampattern and the actual one to form energy focusing in the target spatial sector under the constraint of uniform power distribution, controling sidelobe level, forming nulls in the interference orientation. Hence defining a new matrix $\boldsymbol{H_k}=\boldsymbol{w}_k\boldsymbol{w}_k^{\mathrm{H}}$, the optimization problem is formulated as:

$$\begin{aligned}&\min_{\boldsymbol{H}_1,\cdots,\boldsymbol{H}_K}\ \max_{\theta,\varphi}\left\|P_d(\theta,\varphi)-\sum_{k=1}^{K}Tr\left\{\boldsymbol{a}_k(\theta,\phi)\boldsymbol{a}_k(\theta,\phi)^H\boldsymbol{H}_k\right\}\right\|\\&s.t.\ \sum_{k=1}^{K}diag\{\boldsymbol{H}_k\}=\frac{E}{M_t N_t}\boldsymbol{I}_{M_t N\times 1}\\&Tr(H_k\boldsymbol{a}_k(\theta,\phi)\boldsymbol{a}_k(\theta,\phi)^H)=Tr(\boldsymbol{H}_{k+K/2}\boldsymbol{a}_{k+K/2}(\theta,\phi)\boldsymbol{a}_{k+K/2}(\theta,\phi)^H)\\&\sum_{k=1}^{K}Tr\left\{\boldsymbol{a}_k(\theta_i,\phi_i)\boldsymbol{a}_k(\theta_i,\phi_i)^H\boldsymbol{H}_k\right\}=0\quad i=1,2\cdots L\\&\sum_{k=1}^{K}Tr\left\{\boldsymbol{a}_k(\theta,\phi)\boldsymbol{a}_k(\theta,\phi)^H\boldsymbol{H}_k\right\}=0\quad \theta\in\bar{\Theta},\phi\in\bar{\Phi}\\&\boldsymbol{H}_k\geq 0\quad k=1,\ldots,K\end{aligned} \tag{11}$$

where $P_d(\theta,\phi)$ is the desired beampattern, E is the total power, $Tr(\bullet)$ denotes the trace of a matrix, $diag(\bullet)$ denotes the diagonal of a square matrix. $\boldsymbol{H}_k\geq 0$ means that $\boldsymbol{H}_k$ is positive semidefinite. The convex optimization problem (11) is solved using semidefinite programming（SDP）[10]. After denotes the optimal solution $\boldsymbol{H}_k^{opt}$, the actual transmit matrix $\boldsymbol{w}_k$ need to be solved. We can achieve $\boldsymbol{w}_k$ directly if $\boldsymbol{H}_k^{opt}$ is rank one. Otherwise $\boldsymbol{w}_k$ can be solved by randomization techniques [11] from $\boldsymbol{H}_k^{opt}$.

4 Unitary ESPRIT

Reconstruct the two matrix $\boldsymbol{y_k}$ and $\boldsymbol{y_i}$ from section III, we can achieve a new matrix as

$$\boldsymbol{Y}=[\boldsymbol{y_i}\ \ \boldsymbol{y_k}] \tag{12}$$

The first step is defining a specific unitary matrix as

$$\boldsymbol{U}_{2n}=\frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{I}_n & \mathrm{j}\boldsymbol{I}_n\\ \boldsymbol{\Pi}_n & -\mathrm{j}\boldsymbol{\Pi}_n\end{pmatrix}$$

$$\boldsymbol{U}_{2n+1}=\frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{I}_n & \boldsymbol{0} & \mathrm{j}\boldsymbol{I}_n\\ \boldsymbol{0}^{\mathrm{T}} & \sqrt{2} & \boldsymbol{0}^{\mathrm{T}}\\ \boldsymbol{\Pi}_n & \boldsymbol{0} & -\mathrm{j}\boldsymbol{\Pi}_n\end{pmatrix} \tag{13}$$

where $\boldsymbol{\Pi}$ denotes the exchange matrix containing ones on its main anti-diagonal and zeros elsewhere. Then we transform the complex-valued manifold to a real-valued one as

$$\boldsymbol{H}=(\boldsymbol{U}_M^{\mathrm{H}}\otimes\boldsymbol{U}_N^{\mathrm{H}})\boldsymbol{Y} \tag{14}$$

We proceed eigenvalue decomposition of covariance matrix of the new matrix $[\mathrm{Re}(\boldsymbol{H}),\mathrm{Im}(\boldsymbol{H})]$, and then achieve a signal vector $\boldsymbol{E}_s$ which consists of the large eigenvalues. Defining the selection matrix

$$\boldsymbol{J}_1=\left\lfloor \boldsymbol{I}_{(N-1)(N-1)}\ \boldsymbol{0}_{(N-1)\times 1}\right\rfloor \quad \boldsymbol{J}_2=\left\lfloor \boldsymbol{0}_{(N-1)\times 1}\ \boldsymbol{I}_{(N-1)(N-1)}\right\rfloor \tag{15}$$

Then achieve $\boldsymbol{U}_{N-1}^{H}\boldsymbol{J}_2\boldsymbol{U}_N=(\boldsymbol{U}_{N-1}^{H}\boldsymbol{J}_1\boldsymbol{U}_N)^*$ from $\boldsymbol{U}_N\boldsymbol{U}_N^{H}=\boldsymbol{I}$, and defining a group of new matrix

$$\boldsymbol{\vartheta}_\mu=(\boldsymbol{W}_{\mu 1}\times\boldsymbol{E}_s)^{-1}\times\boldsymbol{W}_{\mu 2}\times\boldsymbol{E}_s \quad \boldsymbol{\vartheta}_\nu=(\boldsymbol{W}_{\nu 1}\times\boldsymbol{E}_s)^{-1}\times\boldsymbol{W}_{\nu 2}\times\boldsymbol{E}_s \tag{16}$$

where $\boldsymbol{W}_{\mu 1}=\boldsymbol{I}_M\otimes\mathrm{Re}(\boldsymbol{U}_{N-1}^{\mathrm{H}}\boldsymbol{J}_2\boldsymbol{U}_N)$, $\boldsymbol{W}_{\nu 1}=\mathrm{Re}(\boldsymbol{U}_{M-1}^{\mathrm{H}}\boldsymbol{J}_2\boldsymbol{U}_M)\otimes\boldsymbol{I}_N$, $\boldsymbol{W}_{\mu 2}=\boldsymbol{I}_M\otimes\mathrm{Im}(\boldsymbol{U}_{N-1}^{\mathrm{H}}\boldsymbol{J}_2\boldsymbol{U}_N)$, $\boldsymbol{W}_{\nu 2}=\mathrm{Im}(\boldsymbol{U}_{M-1}^{\mathrm{H}}\boldsymbol{J}_2\boldsymbol{U}_M)\otimes\boldsymbol{I}_N$. Now combine the matrix $\boldsymbol{\vartheta}_\mu$ and $\boldsymbol{\vartheta}_\nu$ into a new matrix $\boldsymbol{P}=\boldsymbol{\vartheta}_\mu+\mathrm{j}\boldsymbol{\vartheta}_\nu$, then a diagonal matrix $\boldsymbol{\Sigma}$ is obtained which is composed of the eigenvalues of the matrix $\boldsymbol{P}$. Finally the estimation of θ,ϕ is achieved as

$$\hat{\theta}=\arcsin\sqrt{\boldsymbol{u}^2+\boldsymbol{v}^2} \quad \hat{\varphi}=-\arctan(\frac{\boldsymbol{v}}{\boldsymbol{u}}) \tag{17}$$

where $\boldsymbol{u}=2/\pi\times\tan(\mathrm{Re}(\boldsymbol{\Sigma})),\boldsymbol{v}=2/\pi\times\tan(\mathrm{Im}(\boldsymbol{\Sigma}))$.

5 Simulation results

In our simulations, we assume a 5×5 URA with $d_m=d_n=\lambda/2$, where λ is the wavelength. The target spatial range is defined by $\Theta=[-60,-20]$, $\Phi=[20,60]$. The 2D transmit array is divided into K=2 subarrays that are fully overlapped [12] and each of them consists of 24 antennas. The noise is complex Gaussian with zero mean. There is an interference located in $\theta_i=-10^\circ$, $\phi_i=80^\circ$. Fig.1 shows the overlapped transmit array.

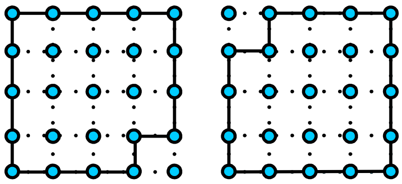

Fig.1: Fully overlapped subarrays of 5×5 URA when K=2

The optimization transmit beampattern obtained by solving (11) is shown in Fig.2 and Fig.3. It can be seen from the figure that the transmit power is focused within the desired 2D target spatial range with 30° width at each side of the mainlobe in the elevation domain and 25° at each side of the mainlobe in the azimuth domain. The sidelobes is constrained below -25dB. And it can form nulls in the interference location.

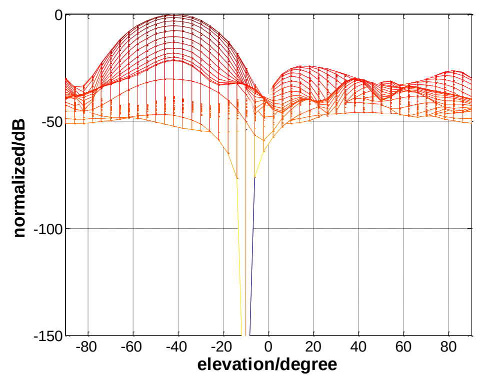

Fig.2: Transmit beampattern: elevation aspect

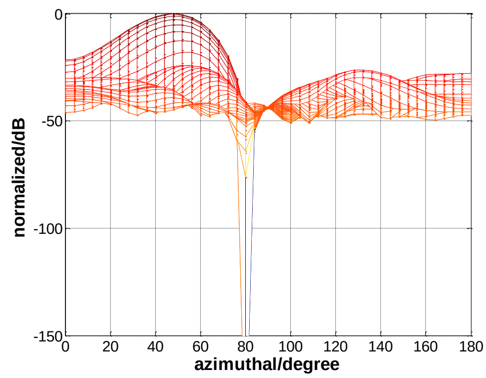

Fig.3: Transmit beampattern: azimuthal aspect

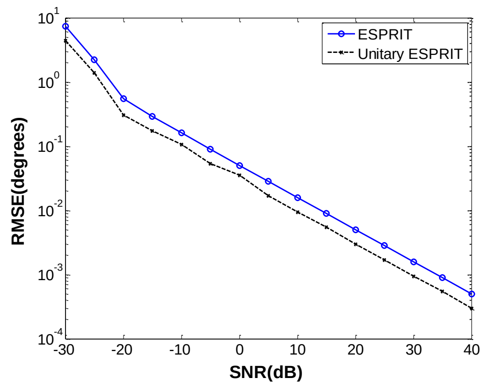

Fig.4 : RMSE over 500 trials, from -30 to 30 dB.

In the second example, we compare the DOA performance of Unitary ESPRIT and traditional ESPRIT on the basis of transmit beampattern design. Two targets are assumed to be located in the far-field at the elevation direction $\theta_1=-45^\circ$ and $\theta_2=-35^\circ$, and the azimuth direction $\phi_1=50^\circ$ and $\phi_2=40^\circ$. The Monte Carlo trials is 500. To evaluate the DOA performance, we consider the root mean square error (RMSE), which is calculated as

$$RMSE=\sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(\frac{1}{L}\sum_{l=1}^{L}\left\|\hat{\beta}_l-\beta_l\right\|^2\right)} \tag{18}$$

where M is the number of Monte Carlo trials, $\beta=[\theta_l\quad\phi_l]$ are the target locations, and $\hat{\beta}=[\hat{\theta}_l \quad \hat{\phi}_l]$ are their estimations. As can be seen in Fig.4, the Unitary ESPRIT enjoys significant performance benefits compared to the traditional ESPRIT in RMSE.

6 Conclusion

The transmit beamspace design for search-free DOA estimation of two dimensional Hybrid MIMO phased-array radar is considered. In addition to the rotational invariance property design, we also consider the transmit beampattern under traditional requests, such as unit power distribution, sidelobe controling, nulls forming. Our simulation results show the improved DOA performance for the proposed method as comparing to the traditional ESPRIT.

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