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The Well-Posedness and Regularity of a Batch Arrival Queue

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Abstract. In this paper, the solution of a batch arrival queue with an additional service channel under N policy is investigat-ed. By using the method of functional analysis, especially, the linear operator theory and the C_0 semigroup theory on Banach space, we prove the well-posedness of the system, and show the existence of positive solution.

Introduction

Recently, Medhi [1] considered an M/G/1 queueing system with a optional service channel, where a unit may depart from the system either after first essential service (FES) with probability $1-\theta$ or at the end of the FES may immediately go for a second phase of service (SPS) with probability $\theta(0\le\theta\le 1)$ In fact some aspects of this model was first studied by Madan [2]. Also he cited some important applications of this model in many real life situations.

In Ref. [3], Choudhury and Paul first obtain the steady state queue size distribution at a random epoch as a gener-alization of result obtained in Medhi [1]. Next they obtain the probability generating function (PGF) of the departure point queue size distribution. Further, they demonstrate the existence of stochastic decomposition property for the queue size distributions. Finally they obtain the mean queue size along with a numerical illustration.

In this paper we consider an $M^{x/}G/1$ queueing system where the arrival occur according to a Compound Poisson process with arrival size random variable X in a two phases heterogeneous service system. The server is turned off each time if system becomes empty. As soon as the queue size is at least N (threshold) ($\geq$1) the server is turned on and begins to serve first phase of essential service (FES) for all the units. Assuming that the service times B_1; B_2 of two channels are mutually independent of each other having general law with distribution function $B_i(x)$; i = 1; 2 (denoting FES and SPS channels respectively). Let us now define the following notations:

l : group arrival rate; X : group size random variable;

$$a_k : \mathrm{Prob}\{X=k\}, k=1,2,\ldots; \sum_{k=1}^{\infty} a_k = 1, \mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)}, i=1,2.$$

It is also assumed that $B_i(0)=0, B_i(\infty)=1, B_i(x), i=1,2$ are continuous at x=0.

The system of differential equations associated with the model as follows([3]):

$$
\begin{cases}
\frac{dR_0(t)}{dt} = -\lambda R_0(t) + \int_0^{+\infty} \mu_2(x) Q_1(x,t)dx + (1-\theta)\int_0^{+\infty} \mu_1(x) P_1(x,t)dx \\
\frac{dR_k(t)}{dt} = -\lambda R_k(t) + \lambda \sum_{i=1}^{k} a_i R_{k-i}(t), k = 1,2...N-1 \\
\frac{\partial P_1(x,t)}{\partial t} + \frac{\partial P_1(x,t)}{\partial x} = -[\lambda + \mu_1(x)] P_1(x,t) \\
\frac{\partial P_n(x,t)}{\partial t} + \frac{\partial P_n(x,t)}{\partial x} = -[\lambda + \mu_1(x)] P_n(x,t) + \lambda \sum_{n=1}^{n-1} a_i P_{n-i}(x,t), n \geq 2 \\
\frac{\partial Q_1(x,t)}{\partial t} + \frac{\partial Q_1(x,t)}{\partial x} = -[\lambda + \mu_2(x)] Q_1(x,t) \\
\frac{\partial Q_n(x,t)}{\partial t} + \frac{\partial Q_n(x,t)}{\partial x} = -[\lambda + \mu_2(x)] Q_n(x,t) + \lambda \sum_{n=1}^{n-1} a_i Q_{n-i}(x,t), n \geq 2
\end{cases}
\tag{1}
$$

with the boundary conditions:

$$
\begin{cases}
P_n(0,t) = (1-\theta)\int_0^{+\infty} \mu_1(x) P_{n+1}(x,t)dx + \int_0^{+\infty} \mu_2(x) Q_{n+1}(x,t)dx, n = 1,2,\dots,N-1 \\
P_n(0,t) = (1-\theta)\int_0^{+\infty} \mu_1(x) P_{n+1}(x,t)dx + \int_0^{+\infty} \mu_2(x) Q_{n+1}(x,t)dx + \lambda \sum_{i=0}^{n-1} a_{n-i} R_i(t), n \geq N, \\
Q_n(0,t) = \theta \int_0^{+\infty} \mu_1(x) P_n(x,t)dx, n \geq 1
\end{cases}
\tag{2}
$$

Equations (1)(2) should be solved together with the normal-izing condition

$$
\sum_{k=0}^{N-1} R_k(t) + \sum_{n=1}^{\infty} \left[\int_0^{+\infty} P_n(x,t)dx + \int_0^{+\infty} Q_n(x,t)dx \right] = 1
$$

and an initial conditions $R_0(0) = 1$:

This paper is organized as follow. In section 2, we shall prove the well-posedness of the system, by using the C_0 semigroup theory. In section 3, we show that the kinetic operator of system generates a positive C_0 semigroup, and study the regularity of the system.

The Well-Posedness of the System

In the following, we always denote by R; R^+; N^+; the real number set, the non-negative real number set, the positive integer number set, respectively. Let $X = R^N \times L^1(R^+ \times N^+) \times L^1(R^+ \times N^+)$ equipped with the norm

$$
\left\| (R_k, P_n(x), Q_n(x)) \right\| = \sum_{k=0}^{N-1} |R_k| + \sum_{n=1}^{\infty} \left[\|P_n(x)\|_1 + \|Q_n(x)\|_1 \right]
$$

for $P = (R_k, P_n(x), Q_n(x)) \in X$. It is easily to see that X is a Banach space.

We define the operator $A = A_1 + B$ by

$$
A_1 \begin{pmatrix} R_0 \\ R_k \\ P_n(x) \\ Q_n(x) \end{pmatrix} = \begin{pmatrix} -\lambda_0 + \int_0^{+\infty} \mu_2(x) Q_1(x)dx + (1-\theta)\int_0^{+\infty} \mu_1(x) P_1(x)dx \\ -\lambda R_k, 1 \leq k \leq N-1 \\ -P_n'(x) - [\lambda + \mu_1(x)] P_n(x) \\ -Q_n'(x) - [\lambda + \mu_2(x)] Q_n(x) \end{pmatrix}
$$

$$B\begin{pmatrix} R_0 \\ R_k \\ P_1(x) \\ P_n(x) \\ Q_1(x) \\ Q_n(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda\sum_{i=1}^{k} a_i R_{k-i}, 1 \le k \le N-1 \\ 0 \\ \lambda\sum_{i=1}^{n-1} a_i P_{n-i}(x), n \ge 2 \\ 0 \\ \lambda\sum_{i=1}^{n-1} a_i Q_{n-i}(x), n \ge 2 \end{pmatrix} \quad (3)$$

with the domain

$$D(B) = X, D(A) = D(A_1), and$$

$$D(A) = \{R_k, P_n(x), Q_n(x) \in X : P_n'(x), \mu_1 P_n(x), Q_n'(x), \mu_2 Q_n(x) \in L^1(R^+),$$
$$P_n(0) = (1-\theta)\int_0^{+\infty} \mu_1(x) P_{n+1}(x)dx + \int_0^{+\infty} \mu_2(x) Q_{n+1}(x)dx, 1 \le n \le N-1$$
$$P_n(0) = (1-\theta)\int_0^{+\infty} \mu_1(x) P_{n+1}(x)dx + \int_0^{+\infty} \mu_2(x) Q_{n+1}(x)dx + \lambda\sum_{i=0}^{N-1} a_{n-i} R_i, n \ge N$$
$$Q_n(0) = \theta\int_0^{+\infty} \mu_1(x) P_n(x)dx, n \ge 1\}$$

$P_n(x), Q_n(x), n \ge 1$ are absolutly continuous.

Then the equation system(1)(2)can be rewritten as an abstract Cauchy problem on X:

$$\begin{cases} \dfrac{dP(t)}{dt} = AP(t) \\ P(0) = \tilde{P}_0 \end{cases} \quad (4)$$

where $P_t = (R_k, P_n(x,t), Q_n(x,t)), \tilde{P}_0 = (1,0,0,...)$

Theorem 2.1: A_1 is a linear closed and densely defined operator on X.

Proof of Theorem 2.1 is a direct verification, so we omit the details.

Let X be the dual space of X, and A_1 be the dual operator of A_1, then

$$X^* = R^N \times L^\infty(R^+ \times N^+) \times L^\infty(R^+ \times N^+).$$

For any $P = (R_k, P_n(x), Q_n(x)) \in D(A_1), Q = (r_k, r_n(x), q_n(x)) \in X^*$ from (A_1P; Q) = (P; A_1Q), we can obtain

$$A_1\begin{pmatrix} r_0 \\ r_n \\ p_1(x) \\ p_n(x) \\ q_1(x) \\ q_n(x) \end{pmatrix} = \begin{pmatrix} -\lambda r_0 + \lambda \sum_{n=N}^{\infty} a_n p_n(0) \\ -\lambda r_k + \lambda \sum_{n=N}^{\infty} a_{n-k} p_n(0) \\ p_1'(x) - [\lambda + \mu_1(x)] p_1(x) + [(1-\theta) r_0 + \theta q_1(0)] \mu_1(x) \\ p_n'(x) - [\lambda + \mu_1(x)] p_n(x) + [(1-\theta) p_{n-1}(x) + \theta q_n(0)] \mu_1(x) \\ q_1'(x) - [\lambda + \mu_2(x)] q_1(x) + r_0 \mu_2(x) \\ q_n'(x) - [\lambda + \mu_2(x)] q_n(x) + p_{n-1}(0) \mu_2(x) \end{pmatrix} \tag{5}$$

Where $1 \le k \le N-1, n \ge 2$, and with the domain

$$D(A_1^*) = \{(r_k, p_n(x), q_n(x)) \in X^* : p_n'(x), \mu_1(x) p_n(x), q_n'(x), \mu_2(x) q_n(x) \in L^\infty(R^+), n \ge 1\}$$

Theorem 2.2 1 is not an eigenvalue of A_1^*.

Proof Let $Q = (r_k, p_n(x), q_n(x)) \in X^*, A_1^* Q = Q$, *i.e.*

$$\begin{cases} -\lambda r_0 + \lambda \sum_{n=N}^{\infty} a_n p_n(0) = r_0 \\ -\lambda r_k + \lambda \sum_{n=N}^{\infty} a_{n-k} p_n(0) = r_k, 1 \le k \le N-1 \\ p_1'(x) - [\lambda + \mu_1(x)] p_1(x) + [(1-\theta) r_0 + \theta q_1(0)] \mu_1(x) = p_1(x) \\ p_n'(x) - [\lambda + \mu_1(x)] p_n(x) + [(1-\theta) p_{n-1}(0) + \theta q_n(0)] \mu_1(x) = p_n(x) \\ q_1'(x) - [\lambda + \beta(x)] q_1(x) + r_0 \mu_2(x) = q_1(x) \\ q_n'(x) - [\lambda + \beta(x)] q_n(x) + p_{n-1} \mu_2(x) = q_n(x) \end{cases} \tag{6}$$

Where n≥2, from (6) we get

$$\begin{cases} p_1(0) = [(1-\theta) r_0 + \theta q_1(0)] \int_0^{+\infty} \mu_1(u) e^{-\int_0^u [1+\lambda+\mu_1(s)] ds} du \\ p_n(0) = [(1-\theta) p_{n-1}(0) + \theta q_n(0)] \int_0^{+\infty} \mu_1(u) e^{-\int_0^u [1+\lambda+\mu_1(s)] ds} du, n \ge 2 \\ q_1(0) = r_0 \int_0^{+\infty} \mu_2(u) e^{-\int_0^u [1+\lambda+\mu_2(s)] ds} du \\ q_n(0) = p_{n-1}(0) \int_0^{+\infty} \mu_2(u) e^{-\int_0^u [1+\lambda+\mu_2(s)] ds} du \end{cases} \tag{7}$$

Where n≥2, Since

$$\int_0^{+\infty} \mu_i(u) e^{-\int_0^u [1+\lambda+\mu_i(s)] ds} du = c_i \in (0,1), i = 1,2 \tag{8}$$

$$p_1(0) = [(1-\theta) + \theta c_2] c_1 r_0, p_n(0) = \{[(1-\theta) + \theta c_2] c_1\}^n r_0, n \ge 2 \tag{9}$$

If $r_0 \neq 0$,then from (6) we get $\lambda \sum_{n=N}^{\infty} a_n \left\{\left[(1-\theta)+\theta c_2\right]c_1\right\}^n = 1+\lambda$

Observing $\left[(1-\theta)+\theta c_2\right]c_1 \in (0,1), a_n \geq 0, \sum_{n=1}^{\infty} a_n = 1,$

we obtain $1+\lambda = \lambda \sum_{n=N}^{\infty} a_n \left\{\left[(1-\theta)+\theta c_2\right]c_1\right\}^n \leq \lambda \sum_{n=N}^{\infty} a_n \leq \lambda$

This contradition means $r_0 = 0$,hence $r_k = p_n(0) = q_n(0) = 0$,Further more we get qn(0) = 0: Furthermore we get $p_n(x) \equiv 0$; $q_n(x) \equiv 0$; n≥1. Hence Q = 0 and 1 is not a eigenvalue of A_1^*.

Theorem 2.3: (1) A is a dissipative operator on X.

(2) The operator A_1 generates a C_0 semigroup of contrac-tion.

Proof Firstly, we will prove that A is a dissipative operator on X. In fact, for any

$P = (R_k; P_n(x); Q_n(x)) \in D(A)$;we define $Q = (r_k; p_n(x); q_n(x)) \in X^*$,

Where

$r_k = \|P\| \operatorname{sgn}(R_k)$; $p_n(x) = \|P\| \operatorname{sgn}(P_n(x))$; $q_n(x) = \|P\| \operatorname{sgn}(Q_n(x))$;then $(P; Q) = \|P\| \|Q\|$ In addition, we have

$$(AP,Q) = \|P\| \Big\{ \Big[-\lambda R_0 + (1-\theta)\int_0^{+\infty} \mu_1(x) P_1(x)dx + \int_0^{+\infty} \mu_2(x) Q_1(x)dx + \Big] \operatorname{sgn}(R_0)$$

$$+ \sum_{k=1}^{N-1} \Big[-\lambda R_k + \lambda \sum_{i=1}^{k} a_i R_{n-i} \Big] \operatorname{sgn}(R_k) - \int_0^{+\infty} \big[P_1'(x) + (\lambda + \mu_1(x)) P_1(x) \big] \operatorname{sgn}(P_1(x)) dx$$

$$- \sum_{n=2}^{\infty} \int_0^{+\infty} \Big[P_n'(x) + (\lambda + \mu_1(x)) P_n(x) - \lambda \sum_{i=1}^{n-1} a_i P_{n-i}(x) \Big] \operatorname{sgn}(P_n(x)) dx$$

$$- \int_0^{+\infty} \big[Q_1'(x) + (\lambda + \mu_2(x)) Q_1(x) \big] \operatorname{sgn}(Q_1(x)) dx$$

$$- \sum_{n=2}^{\infty} \int_0^{+\infty} \Big[Q_n'(x) + (\lambda + \mu_2(x)) Q_n(x) - \lambda \sum_{i=1}^{n-1} a_i Q_{n-i}(x) \Big] \operatorname{sgn}(Q_n(x)) dx \Big\}$$

$$\leq \|P\| \Big\{ -\lambda |R_0| + (1-\theta)\int_0^{+\infty} \mu_1(x) |P_1(x)| dx + \int_0^{+\infty} \mu_2(x) |Q_1(x)| dx$$

$$+ \sum_{k=1}^{N-1} \Big[-\lambda |R_k| + \lambda \sum_{i=1}^{k} a_i |R_{n-i}| \Big] + \sum_{n=1}^{N-1} \Big[(1-\theta)\int_0^{+\infty} \mu_1(x) |P_{n+1}(x)| dx + \int_0^{+\infty} \mu_2(x) |Q_{n+1}(x)| dx \Big]$$

$$+ \sum_{n=N}^{\infty} \Big[(1-\theta)\int_0^{+\infty} \mu_1(x) |P_{n+1}(x)| dx + \int_0^{+\infty} \mu_2(x) |Q_{n+1}(x)| dx \Big] + \lambda \sum_{n=N}^{\infty} \sum_{i=1}^{N-1} a_{n-i} R_i$$

$$-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\lambda+\mu_1(x)\right]\left|P_n(x)\right|dx+\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}\left|P_{n-i}(x)\right|dx+\sum_{n=1}^{\infty}\theta\int_0^{+\infty}\mu_1(x)\left|P_n(x)\right|dx$$

$$-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\lambda+\mu_2(x)\right]\left|Q_n(x)\right|dx+\lambda\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}\left|Q_{n-i}(x)\right|dx\Bigg\}=0.$$

Therefore, A is dissipative. Observing $(A_1P,Q)\leq (AP,Q)$; we known that A_1 is dissipative and hence $R(I-A_1)$ is a closed subspace of X.

Furthermore, we have $R(I-A_1) = X$. If it is not true, then there exists a $Q\in X^*$; such that for any $F\in R(I-A_1)$;$(F,Q) = 0$: Hence, for any $P\in D(A_1)$; $((I-A_1)P,Q) =0$; i.e., for any $P\in D(A_1)$; $(P, (I-A_1)Q) = 0$.Since $D(A_1)$ is dense in X, thus $A_1^*Q = Q$; this means 1 is a eigenvalue of A_1^*, which contradicts with Theorem 2.2. Hence $R(I- A_1) = X$. So the Lumer-Philips Theorem([4]) asserts that A_1 generates a C_0 semigroup of contraction.

Theorem 2.4: The operator A generates a C_0 semigroup on X. The system (4) is well-posed.

Proof Obviously, B is a bounded linear operator on X; using the perturbation theory of semigroup ([4]), we know that the operator A generates a C_0 semigroup on X. Therefore, the system (4) is well-possed.

The Regularity of Solution

In the following, let X be a real Banach space. The concepts and theory of Banach lattice, positive cone and positive semigroup can be referred to Ref. [5].

Definition 3.1: ([5]) Let X be a Banach lattice, X_+ be a positive cone of X and A be a linear operator on X. Denote $G(x)=\left\{\varphi\in X_+^*:\|\varphi\|\leq 1,(x,\varphi)=\|x_+\|^2\right\}$. If, for any $x\in D(A)$; there exists $\varphi\in G(x)$; such that $(Ax,\varphi\leq 0$; then A is called a dispersive operator.

From Ref. [5], we know that the following result is true. Lemma 3.1: Let X be a Banach lattice and A be a linearclosed defined operator on X. Then A generates a positive contractive semigroup if and only if A is a dispersive operator and $R(I-A) = X$.

Theorem 3.1: (1) A is a dissipative operator on X.

(2) The operator A generates a positive C_0 contractive semigroup on X.

Proof It is well known that X is a Banach lattice. According to Lemma 3.1, it is sufficient to prove that A is a dispersiveoperator. For any $P = (R_k,P_n(x),Q_n(x))$I D(A); we choose $Q = (r_k,p_n(x),q_n(x))\in X$, where $r_k=\|P\|\ \mathrm{sgn}_+(R_k)$; $p_n(x)=\|P\|\ \mathrm{sgn}_+(P_n(x)), q_n(x)=\|P\|\,\mathrm{sgn}_+(Q_n(x))$,and if a > 0, then $\mathrm{sgn}_+a = 1$; if a 0, then $\mathrm{sgn}_+a = 0$.

Similar to the proof of Theorem 2.3, a direct verification can show that $(AP,Q)\leq 0$. Observing $Q\in G(P)$; the desired result follows from Lemma 3.1.

The following result studies the regularity of the system.

Theorem 3.2: Let T (t) be a positive contractive semigroup with generator A, then T (t) satisfies positive conserve property, i.e., for any $H_0\in D(A)$ and $H_0> 0$, $\|T(t)H_0\|=\|H_0\|, t\geq 0.$

Proof Since $H_0\in D(A)$ and $H_0> 0$; then $T(t)H_0\in D(A)$ is a classical solution of the system (4). Let$P(t) = (R_k(t),P_n(x; t),Q_n(x,t)) = T(t)H_0> 0$,then P (t) satisfies (1)(2).Note that

$$\frac{d}{dt}\left\|P(t)\right\|=\frac{d}{dt}\left\|T(t)H_0\right\|=\sum_{n=0}^{N-1}\frac{dR_n(t)}{dt}+\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{\partial P_n(x)}{\partial t}dx+\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{\partial Q_n(x,t)}{\partial t}dx$$
,

We get

$$\frac{d}{dt}\left\|P(t)\right\|=-\sum_{n=0}^{N-1}\lambda R_n(t)+\lambda\sum_{n=1}^{N-1}\sum_{i=1}^{n}a_iR_{n-i}(t)+(1-\theta)\int_0^{+\infty}\mu_1(x)P_1(x,t)dx+\int_0^{+\infty}\mu_2(x)Q_1(x,t)dx$$

$$-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\frac{\partial P_n(x,t)}{\partial x}+(\lambda+\mu_1(x))P_n(x,t)\right]dx+\lambda\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}P_{n-i}(x,t)dx$$

$$-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\frac{\partial Q_n(x,t)}{\partial x}+(\lambda+\mu_2(x))q_n(x,t)\right]dx+\lambda\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}Q_{n-i}(x,t)dx$$

$$=-\sum_{n=0}^{N-1}\lambda R_n(t)+\lambda\sum_{n=2}^{N-1}\sum_{i=1}^{n}a_i\int_0^{+\infty}a_iR_{n-i}(t)dt+(1-\theta)\int_0^{+\infty}\mu_1(x)P_1(x,t)dx$$

$$+\int_0^{+\infty}\mu_2(x)Q_1(x,t)dx-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\lambda+\mu_1(x)\right]P_n(x,t)dx+\sum_{n=1}^{\infty}(1-\theta)\int_0^{+\infty}\mu_1(x)P_{n+1}(x,t)dx$$

$$+\sum_{n=1}^{\infty}\int_0^{+\infty}\mu_2(x)Q_{n+1}(x,t)dx+\lambda\sum_{n=N}^{\infty}\sum_{i=0}^{n-1}a_{n-i}R_i(t)+\lambda\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}P_{n-i}(x,t)dx$$

$$+\sum_{n=1}^{\infty}\theta\int_0^{+\infty}\mu_1(x)P_n(x,t)dx-\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\lambda+\mu_2(x)\right]Q_n(x,t)dx+\lambda\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}a_i\int_0^{+\infty}Q_{n-i}(x,t)dx=0.$$

Hence $\left\|P(t)\right\|=\left\|P(0)\right\|=\left\|H_0\right\|$.

References

[1]J. Medhi, A single server Poisson arrival queue with a second optional channel, Queueing Systems, 42(2002), 239-242.

[2]K. C. Madan, An M=G=1 queue with second optional service, Queuing Systems, 34(2000), 37-46.

[3]G. Choudhury, M. Paul, A batch arrival queue with an additional service channel under N policy, Applied Mathematics and Computation,156(2004), 115-130.

[4]A. Pazy, Semigroup of linear operators and applications to partial differ-ential equations, Springer-Verlag, New York, 1983.

[5]J. A. Goldstein, Semigroups of linear operators and applications, Oxford University Press, New York, 1985.