An Algorithm for the Orientation of Complete Bipartite Graphs

Lingqi Zhao1, Mujiangshan Wang2, Xuefei Zhang3, Yuqing Lin4, Shiyong Wang5,*
1Institute of Discrete Mathematics, College of Computer Science and Technology, Inner Mongolia University for Nationalities, Tongliao 028043 PR China
2School of Electrical Engineering and Computer Science, The University of Newcastle NSW 2308, Australia
3School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, PR China
4School of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, PR China
*Corresponding author

Abstract—Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). We consider the problem of orienting the edges of a complete bipartite graph \( K_{n,n} \) so that only two different in-degrees \( a \) and \( b \) occur. An obvious necessary condition for orienting the edges of \( G \) so that only two in-degrees \( a \) and \( b \) occur, is that there exist positive integers \( s \) and \( t \) satisfying \( s + t = |V(G)| \) and \( as + bt = |E(G)| \). In this paper, we show that the necessary condition is also sufficient for a complete bipartite graph\( K_{n,n} \). Furthermore, we give the algorithms of orientations with only two in-degrees of \( K_{n,n} \).

Keywords—complete bipartite graph; orientation; algorithm

1. INTRODUCTION

An orientation \( D = (V(D), A(D)) \) of an undirected graph \( G = (V(G), E(G)) \) is a digraph obtained by replacing each undirected edge \( e \in E \) with an arc from one end vertex of \( e \) to the other. In the oriented problem, we are asked whether \( G \) has an orientation satisfying some conditions. This is a basic problem in combinatorial optimization, and many beautiful results have been produced so far. Chen et al. [3] studied orientations of graphs satisfying the Ore condition. Fukunaga [4] investigated graph orientations with set connectivity requirements. Miao and Lin [7] gave strong orientations of complete bipartite graphs achieving the strong diameter. The main purpose of this paper is to orient \( K_{n,n} \) with \( a \) or \( b \) arrowheads directed towards each vertex.

As a digraph, there is a problem of oriented graphs in practice. Buhler et al. [1] considered the problem of orienting the edges of the \( n \)-dimensional hypercube so that only two in-degrees occur for finding strategies for specific hat guessing games.

Let \( D \) be a digraph, for any \( uv \in A(D) \), we say that \( u \) dominates \( v \) (or \( v \) is dominated by \( u \)) and denote it by \( u \rightarrow v \).

For any \( v \in V(D) \), the in-degree of \( v \) is denoted by \( d^-(v) = |\{u \in V(D) : uv \in A(D)\}| \) and the out-degree of \( v \) is denoted by \( d^+(v) = |\{u \in V(D) : vu \in A(D)\}| \). For disjoint subsets \( X \) and \( Y \) of \( V(D) \), \( X \rightarrow Y \) means that every vertex of \( X \) dominates every vertex of \( Y \), and we define \( [X,Y] = \{xy \in A(D) : x \in X, y \in Y\} \). For graph-theoretical terminology and notation not defined here we follow[2, 6].

In this paper, \( K_{n,n} \) is oriented to a digraph so that only two in-degrees \( a \) and \( b \) occur. For convenience, let \( [a,b] \), be a shorthand for the problem of realizing an orientation of \( K_{n,n} \) whose only in-degrees are \( a \) or \( b \). In Section 2, we give a necessary and sufficient condition such that \([a,b]\) is realizable. In Section 3, we give some specified algorithms to construct the required orientations of \( K_{n,n} \).

II. MAIN RESULTS

Lemma 2.1. Given a positive integer \( n \), let \( K_{n,n} \) be a complete bipartite graph. For \( a,b \in \{0,1,2,\ldots,n\} \), if \([a,b]\) is realizable, then there exist positive integers \( s \) and \( t \) satisfying the following two equations:

\[
\begin{align*}
\begin{cases}
    s + t = 2n, \\
    as + bt = n^2.
\end{cases}
\end{align*}
\]

Proof. Let \([a,b]\) be realizable. Then there exists an oriented graph whose only in-degrees are \( a \) or \( b \). Let the in-degree of \( s \) vertices in \( K_{n,n} \) be \( a \). Then the in-degree of the remaining \((2n-s)\) vertices is \( b \). Therefore, \( s + t = |V(K_{n,n})| = 2n \), and \( as + bt = |E(K_{n,n})| = n^2 \), where \( t = n-s \).

Let \( G \) be a nonoriented graph. For \( U \subseteq V(G) \), denote the number of edges which have their both end-vertices in \( U \) by \( e(U) \).
Lemma 2.2. [5] Given a nonoriented graph $G$ whose vertices are labeled $v_1, v_2, \ldots, v_k$ and to whose vertices are associated non-negative integers $v(v_i), v(v_2), \ldots, v(v_k)$, respectively; then, $G$ is orientable with $v(v_i)$ arrowheads directed toward vertex $v_i$ (for $i \in \{1, 2, \ldots, n\}$) if and only if

$$\sum_{v \in \overrightarrow{E}(G)} v(v) = |E(G)|$$

and

$$e(U) \leq \sum_{v \in U} v(v)$$

for each $U \subseteq V(G)$.

(1)

Lemma 2.3. Given three positive integers $a, b$, and $t$, let $K_{a,b}$ be a complete bipartite graph and let $a, b \in \{0,1,2,\ldots,n\}$ with $a \leq b$. If $a, b, s, t$ satisfy the following two equations:

$$\begin{cases} s + t = 2n, \\ as + bt = n^2. \end{cases}$$

(2)\hspace{1cm}(3)

Then $[a, b]_n$ is realizable.

Proof. Case 1. $s > t$.

By $a \leq b$ and $a, b \in \{0,1,2,\ldots,n\}$, $n-a \geq n-b$ and $n-a, n-b \in \{0,1,2,\ldots,n\}$. We can deduce that

$$(n-a)s + (n-b)t = (n-a)s + (2n-s)(n-b) = 2n^2 - [as + bt]$$

$= 2n^2 - n^2 = n^2$. Then $n-a$ and $n-b$ satisfy the conditions of Lemma 2.3. By Lemma 2.3, $[n-a, n-b]_n$ is realizable. Then $K_{a,b}$ has an orientation $D$ whose only in-degrees are $n-a$ or $n-b$. We consider the digraph $D'$ obtained by reversing all the arcs in $D$. Note that $K_{a,b}$ is $n$-regular. Then $[n-(n-a), n-(n-b)]_n = [a, b]_n$ is realizable.

By Lemmas 2.1, 2.3 and 2.4, we can obtain the following theorem directly.

Theorem 2.5. Given a positive integer $n$, let $K_{a,b}$ be a complete bipartite graph. For $a, b \in \{0,1,2,\ldots,n\}$, $[a, b]_n$ is realizable if and only if there exist positive integers $s$ and $t$ satisfying the following two equations:

$$\begin{cases} s + t = 2n, \\ as + bt = n^2. \end{cases}$$

(4)

Corollary 2.6. Given a positive integer $n$, let $K_{a,b}$ be a complete bipartite graph. For $a, b \in \{0,1,2,\ldots,n\}$, the following results hold:

(a) if $[a, b]_n$ is realizable, then $[n-a, n-b]_n$ is realizable;

(b) $[0, n]_n$ is realizable;

(c) if $n$ is even, then $[\frac{n}{2}, \frac{n}{2}]_n$ is realizable.

III. ORIENTATION ALGORITHMS OF $K_{a,b}$

In Section 2, we have proved that $K_{a,b}$ admits the orientation with only two in-degrees. In this section, we will show how to orient $K_{a,b}$ by specified algorithms. By the proofs of Lemmas 2.3 and 2.4, it is enough to consider the case where $a \leq b$ and $s \leq t$.

Specially, $K_{1,1}$ has an orientation $D$ whose only in-degrees are 0 or 1. Then $[0, 1]_n$ is obviously realizable. In the following, suppose that $n \geq 2$.

If $a = b$, then, by the equations (2) and (3), $a = b = \frac{n}{2}$.

Note that $a$ is an integer and $a = \frac{n}{2}$. Therefore, $n$ is even.

Combining this with the fact that the degree of each vertex in $K_{a,b}$ is $n$, $K_{a,b}$ admits an Euler tour. By the definition of Euler tour, there exists an oriented graph of $K_{a,b}$ whose only in-degree is $\frac{n}{2}$. Then $[\frac{n}{2}, \frac{n}{2}]_n$ is realizable. Next assume that $a < b$. By the equations (2) and (3), we have $s = \frac{m(2b-n)}{b-a}$.
and \( t = \frac{n(n - 2a)}{b - a} \). Combining this with the fact that \( s \) and \( t \) are positive integers, \( a < \frac{n}{2} < b \).

By \( b = \frac{n}{2} \), \( b > n - b \). Since \( s \leq t \) and \( s + t = 2n \), \( n - s \geq 0 \). We can deduce that
\[
n^2 = as + bt = as + b(2n - s) = as + bn + b(n - s) \geq as + bn + n^2 - bn - bn + bs = (a + b - n)s + n^2
\]
Hence \((a + b - n)s \leq 0 \). Combining this with \( s > 0 \), \( a + b \leq n \).

Case 1. \( a + b = n \).

Let \((X,Y)\) be a bipartition of \( K_{n,n} \) with \( X = \{u_1, u_2, \ldots, u_{n-1}\} \) and \( Y = \{v_1, v_2, \ldots, v_{n-1}\} \). By \( a + b = n \),
\[
s = \frac{n(2b - n)}{b - a} = \frac{m(2b - n)}{b - (n - b)} = \frac{m(2b - n)}{2b - n} = s
\]
Combining this with \( s + t = 2n \), \( t = n \). Conversely, if \( s + t = 2n \), \( t = n \). Therefore, \( [0,n] \) is realizable. Next, assume that \( a > 0 \).

For any \( u_i \in X \), orient \( v_{ij}(mod_{n}) \) for each \( i = 1, 2, \ldots, a-1 \). We orient the remaining edges which are incident to \( u_i \) towards \( Y \). Now, we obtain an oriented graph \( D \). The in-degree of each vertex of \( X \) in \( D \) is \( a \). For any \( v_i \in Y \), by the definition of \( D \), \( v_i \rightarrow u_{ij}(mod_{n}) \) for each \( i = 1, 2, \ldots, a-1 \). The out-degree of each vertex of \( Y \) in \( D \) is \( n-a \). Only two in-degrees \( a \) and \( n-a \) occur in \( D \). Set \( b = n-a \). Therefore, \([a,b]\) is realizable. Then we can obtain the following proposition directly.

**Proposition 3.1.** Let \( a < b \) and \( a + b = n \). Then the oriented graph which is obtained by the above method has only two in-degrees \( a \) and \( b \).

Case 2. \( a + b < n \).

In this case, \( s < t \). By \( a + b < n \) and \( a < b \),
\[
n-s = n-n(2b-n) = \frac{(b-a)n(2b-1)}{b-a} = \frac{n(n-a-b)}{b-a} > 0
\]
I.e., \( s < n \). Combining this with \( s + t = 2n \), \( t > n \). Then \( s < n < t \).

First, assume that \( a = 0 \). By \( a + b < n \), \( b < n \). If \( s \geq b \),
\[
n^2 = as + bt = as + b(2n - s) = as + bn + b(n - s) < as + sn + bn - n = n^2 - n
\]
This is a contradiction. So \( s < b \).

Let \((X,Y)\) be a bipartition of \( K_{n,n} \) with \( X = \{u_1, u_2, \ldots, u_{n-1}\} \), \( Y = \{v_1, v_2, \ldots, v_{n-1}\} \), \( Y_1 \cap Y_2 = \emptyset \), and let \( Y_1 = \{v_0, v_1, \ldots, v_{k-1}\} \), \( Y_2 = \{z_0, z_1, \ldots, z_{n-k-1}\} \). Orient
\( Y_1 \rightarrow X \rightarrow Y_2 \). Now, we obtain an oriented graph \( D \) (see Fig.1). The in-degree of each vertex \( u_i \) of \( X \) in \( D \) is \( b \). The in-degree of each vertex \( v_i \) of \( Y_i \) in \( D \) is \( 0 \) and the in-degree of each vertex \( z_j \) of \( Y_2 \) in \( D \) is \( n \). Denote the vertex set \( \{v_0, \ldots, v_{n-1}\} \) as \( Y_1 \) by \( Y_1 \). By \( s < b \), \( |Y_1| \geq 1 \).

![Figure 1. The oriented graph D](image)

**Algorithm 3.1.**

**INPUT:** the above oriented graph \( D \) with three in-degrees 0, \( b \) and \( n \).

0. Set \( l_i = 0 \) for every \( x = 0, 1, \ldots, b-1 \), \( r_j = n \) for every \( y = 0, 1, \ldots, n-1 \), \( i = 0 \), \( j = 0 \).

1. If \( l_i \geq b \), \( i \Rightarrow i+1 \).

2. If \( r_j \leq b \), \( j \Rightarrow j+1 \).

3. Choose \( u \in X \) satisfying \( v_i \rightarrow u \rightarrow z_j \) in \( D \). Reverse \( v_i \rightarrow u \rightarrow z_j \) in \( D \). Obtain \( D' \). \( D' = D' \).

4. Set \( l_i = l_i + 1 \) and \( r_j = r_j - 1 \).

5. If \( i = b-s-1 \) and \( l_i = b \), output \( D \). Otherwise, go to step 1.

**Theorem 3.2.** Let \( a = 0 \) and \( a + b < n \). Then Algorithm 3.1 outputs \( D \) which has only two in-degrees 0 and \( b \).

**Algorithm 3.2**

**INPUT:** the above oriented graph \( D \) with four in-degrees \( a, b, n-a \), \( n-b \).

0. Set \( l_i = n-a \) for every \( x = 0, 1, \ldots, s-1 \), \( r_j = n-b \) for every \( y = 0, 1, \ldots, n-s-1 \), \( i = 0 \), \( j = 0 \).

1. If \( r_j \geq b \), \( j \Rightarrow j+1 \).

2. If \( l_i \leq b \), \( i \Rightarrow i+1 \).

3. Choose \( w \in \{w_1, \ldots, w_{n-s-1}\} \) satisfying \( z_j \rightarrow w \rightarrow v_i \) in \( D \). Reverse \( z_j \rightarrow w \rightarrow v_i \) in \( D \). Obtain \( D' \). \( D = D' \).
4. Set \( r_j := r_j + 1 \) and \( l_j := l_j - 1 \).

5. If \( j = n - s - 1 \) and \( r_j = b \), output \( D \). Otherwise, go to step 1.

**Theorem 3.3.** Let \( 0 < a < b \leq n - s \), \( a \leq s < t \) and \( a + b < n \). Then Algorithm 3.2 outputs \( D \) which has only two in-degrees \( a \) and \( b \).

**Algorithm 3.3.**

**INPUT:** the above oriented graph \( D \) with four in-degrees \( a, b, n, s - b \) and \( s - a \).

0. Set \( l_s := s - a \) for every \( x = 0, 1, \ldots, s - 1 \), \( r_y := n + s - b \) for every \( y = 0, 1, \ldots, n - s - 1 \), \( i := 0 \), \( j := 0 \).

1. If \( i \leq b \), \( i := i + 1 \).

2. If \( r_j \leq b \), \( j := j + 1 \).

3. Choose \( q \in \{ u_i, u_{i-1} \mod s \} \ldots, u_{(i-a) \mod s} \} \) satisfying \( v_i \rightarrow q \rightarrow z_i \) in \( D \). Reverse \( v_i \rightarrow q \rightarrow z_j \) in \( D \). Obtain \( D' \).

4. Set \( i := i + 1 \) and \( r_j := r_j - 1 \).

5. If \( i = s - 1 \) and \( l_i = b \), output \( D \). Otherwise, go to step 1.

**Theorem 3.4.** Let \( 0 < a < b, n - s < b, s < t \) and \( a + b < n \). Then Algorithm 3.3 outputs \( D \) which has only two in-degrees \( a \) and \( b \).

**Algorithm 3.4.**

**INPUT:** the above oriented graph \( D \) with five in-degrees \( a, b, n, a - b, a + b \).

0. Set \( l_s := n - a \) for every \( x = 0, 1, \ldots, a - 1 \), \( r_y := n + s - b - a \) for every \( y = 0, 1, \ldots, b - a - 1 \), \( i := 0 \), \( j := 0 \).

1. If \( r_j \geq b \), \( j := j + 1 \).

2. If \( l_i \leq b \), \( i := i + 1 \).

3. Choose \( w \in \{ w_j, w_{j-1} \mod (y-a), w_{j-2} \mod (y-a), \ldots, w_{j-a+1} \mod (y-a) \} \) satisfying \( z_j \rightarrow w \rightarrow v_i \) in \( D \). Reverse \( z_j \rightarrow w \rightarrow v_j \) in \( D \).

Obtain \( D' \). \( D := D' \).

4. Set \( r_j := r_j + 1 \) and \( l_i := l_i - 1 \).

5. If \( j = n - a - 1 \) and \( r_j = b \), output \( D \). Otherwise, go to step 1.

**Theorem 3.5.** Let \( 0 < a < b, n - s < b, s < t \) and \( a + b < n \). Then Algorithm 3.4 outputs \( D \) which has only two in-degrees \( a \) and \( b \).

**Algorithm 3.5.**

**ACKNOWLEDGEMENTS**

This work is supported by the National Science Foundation of China (61370001, 61261025, 61262018) and the Natural Science Foundation of Inner Mongolia (2014MS0116, 2014MS0110).

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