Asymptotic Regularity and Uniform Attractor for Non-autonomous Viscoelastic Equations with Memory

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Abstract—In this paper, long-time behavior of a class of non-autonomous viscoelastic equations with fading memory is investigated. We establish the existence of a compact uniform attractor together with its structure in $H^1_0(\Omega) \times H^1_0(\Omega) \times L^2_p(R^3;H^1_0(\Omega))$. The compact uniform attractor is bounded in $D(A) \times D(A) \times L^2_p(R^3;D(A))$ and attracts every bounded set of $H^1_0(\Omega) \times H^1_0(\Omega) \times L^2_p(R^3;H^1_0(\Omega))$.

Keywords—non-autonomous wave equations; asymptotic regularity; uniform attractor; memory; viscoelasticity

I. INTRODUCTION

In this paper, we consider the dynamical behavior of the solutions for the following non-autonomous evolutionary equations with a fading memory

$$u_t - \Delta u_t - \Delta u - \Delta u_r(t) - \int_0^\infty \mu(s)\Delta \eta_r(s)ds + f(u) = g, \quad (1)$$

and

$$\eta_r^- = -\eta_r^+ + u_r.$$ 

The problem is supplemented with the boundary condition

$$u(x,t)|_{\partial \Omega} = 0 \quad \text{for all} \quad t \geq \tau, \tau \in R$$

and initial condition

$$u(x,t) = u_r(x,t), u_t(x,t) = \frac{\partial}{\partial t} u_r(x,t) \quad t \leq \tau, \tau \in R.$$ 

Where $\Omega$ is a bounded smooth domain in $R^3$, $g = g(t)$ is a given external time-dependent forcing, $f$ is the critical nonlinearity.

Problem (1) is related to the following equations like

$$u_t - u_{tt} - u_x - u_{xct} = 0,$$

Which appear as a class of nonlinear evolution equations, and that is used to represent the propagation problems of lengthways-wave in nonlinear elastic rods and Ion-sonic of space transformation by weak nonlinear effect (see for instance [1,3]). Since (1) contains terms $\Delta u_r$, it is essentially different from D’Alembert wave equation.

Let us recall some results concerning the problem (1). In [10, 11] etc, authors studied this equations with Dirichlet boundary conditions as $\mu = 0$. Recently, Araújo et al. [5] and M. Conti [4], H. Yassine and A. Abbas [9] studied the well-posedness for this equations. In particular, Qin [8] obtain the existence of uniform attractors as $f = 0$.

Maybe, we could establish the existence of uniform attractors of (1) using the method in [16, 17], but the regularity and structure cannot obtain directly. In this paper, we will apply the techniques introduced in Sun [14] to overcome the difficulty due to the critical nonlinearity, and establish the asymptotic regularity of the solutions. Based on this regularity result, we obtain the asymptotic compactness of the non-autonomous system and prove the existence of a uniform attractor together with its structure in $H^1_0(\Omega) \times H^1_0(\Omega) \times L^2_p(R^3;H^1_0(\Omega))$. It is noteworthy that the compact uniform attractor is bounded in $D(A) \times D(A) \times L^2_p(R^3;D(A))$.

For conveniences, hereafter let $|\cdot|$ be the modular (or absolute value) of $u$ and $\|\cdot\|$ be the norm of $L^p(\Omega)(P > 1)$. Denote $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$ and $\|\cdot\|_{H^{-1}}$ be the norm of $H^{-1}(\Omega).$ Let $(\cdot,\cdot)$ be a Banach space, we denote respectively the inner product and norm of the weighted space $L^2_p(R^3;\nu)$ by

$$\langle \phi, \psi \rangle_{\nu,\nu} = \int_0^\infty \mu(s)\langle \phi(s), \psi(s) \rangle_{\nu}, ds$$

and

$$\|\psi\|_{\nu,\nu} = \int_0^\infty \mu(s)\|\phi(s)\|_{\nu}^2 ds.$$
Denote $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, and for $r \in R$, let $\varepsilon_r = D(A^{r}^{-})$ and $\|\|$ be the norm of $\varepsilon_r$. We also define the system state space for $(u,u_t,\eta)$ as $H_r$, together with a dense subspace $M$:

$$H_r = \varepsilon_r \times \varepsilon_r \times L^p_0(R;\varepsilon_r),$$

$$M = D(A) \times D(A) \times (L^p_0(R;D(A)) \cap H^2_0(R;H^1_0(\Omega))).$$

We also define the norm of the product space $rH$ as follows

$$\|rH\| = \|(u,v,\eta)\|_{rH} = \frac{1}{2}(\|v\|^2 + \|v\|^2 + \|\eta\|^2).$$

for any $z = (u,v,\eta) \in H_r$.

Let $C$ be an arbitrarily positive constant, which may be differential from line to line, even in the same line.

For the memory kernel $\mu(s)$, we assume the following hypotheses: for all $s \in R^+$ and some $0 < \delta < 1$

$$\mu \in C^1(R^+) \cap L^p(R^+), \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0,$$

$$\mu'(s) + \delta \mu(s) \leq 0 \quad (2)$$

We introduce a new variable of the system,

$$\eta = \eta' (x,s) := u(x,t) - u(x,t - s), \quad s \in R^+,$$

which will be ruled by a supplementary equation. Denoting

$$\eta' = \frac{\partial}{\partial t} \eta', \quad \eta'' = \frac{\partial}{\partial s} \eta'.
$$

Then the following estimate holds (see [17])

$$\left\langle \eta',\eta'' \right\rangle_{\mu,r} \geq \frac{\delta}{2} \|\eta'\|^2_{\mu,r}. \quad (5)$$

The past history $u_r(\tau - s)$ of the variable $u$ satisfies the condition as follows: there exist two positive constants $\varepsilon$ and $\kappa$ such that

$$\int_{\tau - \varepsilon}^{\tau} e^{-\kappa(s)} \|u_r(\tau - s)\| ds \leq \varepsilon. \quad (6)$$

The nonlinearity $f \in C^1(R,R)$, fulfills $f(0) = 0$ satisfies the following decomposition

$$|f'(s)| \leq c(1 + |s|^p) \quad \text{for all } s \in R \quad (7)$$

and

$$\lim \inf_{s \to \infty} \frac{f(s)}{s} \geq -\lambda,$$

for any $s \in R$, where $c, \lambda$ are positive constants and $\lambda$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ with the Dirichlet boundary condition.

Calling $F(s) = \int_0^s f(y) dy$. Notice that by (8), the following inequalities hold for some $0 < \lambda < \lambda_1$ and $c_0 \geq 0$

$$2\int_0^s f(u)u \geq 2\int_0^s F(u) - \lambda \|u\|^2 - c_0 \quad (9)$$

For the time-dependent forcing $g$, we assume the following hypotheses: $g \in L^p_0(R;L^p(\Omega))$ (translation bounded in $L^p_{u(x)}(R;L^p(\Omega))$), and with the norm

$$\|g\|^2_p = \sup_{\tau \in R} \int_{\tau}^{\tau + 1} \|g(s)\|^2 ds < \infty.$$ 

II. PRELIMINARIES

We will complete our task exploiting the transitivity property of exponential attraction [15], that we recall below for the readers convenience.

**Lemma 2.1.** [15] Let $(H,d)$ be an abstract metric space, $U(t;\tau)$ a Lipschitz continuous dynamical process in $H$, $i.e.$

$$U(t + \tau, \tau)z - U(t + \tau, \tau)v \leq L_0 e^{-\nu \tau} \|z - v\|,$$

for appropriate constants $\nu_0 \geq 0$ and $L_0 \geq 0$ which are independent of $z, \tau$ and $t$. We further assume that there exist three subsets $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \subset H$ such that

$$\text{dist}_{\mu}(U(t + \tau, \tau)\mathcal{K}_1, \mathcal{K}_2) \leq L_0 e^{-\nu \tau},$$

$$\text{dist}_{\mu}(U(t + \tau, \tau)\mathcal{K}_2, \mathcal{K}_3) \leq L_0 e^{-\nu \tau},$$

for some $\nu, \nu_2 \geq 0$ and $L_1, L_2 \geq 0$. Then it follows that

$$\text{dist}_{\mu}(U(t + \tau, \tau)\mathcal{K}_1, \mathcal{K}_3) \leq L e^{-\nu \tau},$$

where $\nu = \frac{\nu_1 \nu_2}{\nu_1 + \nu_1 + \nu_2}$ and $L = L_1 L_2 + L_2$.

**Lemma 2.2.** [12] Let $X \subseteq H \subseteq Y$ be Banach spaces, with $X$ reflexive. Suppose that $u_n$ is a sequence that is uniformly bounded in $L^2(0,T;X)$ and $\frac{du_n}{dt}$ is uniformly bounded in $L^p(0,T;Y)$, for some $p > 1$. Then there is a subsequence of $u_n$ that converges strongly in $L^2(0,T;H)$. 

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III. UNIFORM ATTRACTOR IN $H_0$

Throughout the paper, we assume $g_0 \in L^2_0(R;L^2(\Omega))$ and $f \in \Sigma$ is the hull of $g_0$ in $L^2_{loc}(R;L^2(\Omega))$ and $\Sigma$. Assume further that (2)-(3) and (6)-(8).

A. The Well-Posedness

By the standard Faedo-Galerkin methods, it easy to obtain the following result.

**Lemma 3.1.** For any $T > 0$ and $z_0 = (u_0, v_0, \eta_0) \in H_0$, problem (1.1) admits a unique weak solution $z(t) = (u(t), v(t), \eta(t)) \in C([t, T], H_0)$, satisfying

$$u \in L^\infty(\tau^*;H^1_0(\Omega)), u_0 \in L^2(R;H^2_0(\Omega)),
\quad u_0 \in L^2([\tau, \tau^*];H^1_0(\Omega)), \eta \in L^\infty(R;L^2_0(R^*;H^1_0(\Omega)))$$

The proof of Lemma 3.1 is similar to that of Theorem 2.1 of Araújo et al. [5] and hence is omitted.

Form Lemma 3.1 above, for each $g \in L^2_0(R;L^2(\Omega))$ we define a process $U_g(t, \tau) : H_0 \to H_0,
\quad z_0 = (u_0, v_0, \eta_0) \to (u(t), v(t), \eta(t)) = U_g(t, \tau)z_0$.

B. Dissipativity

First of all, we can obtain the following theorem from [4]:

**Theorem 3.2.** There exists a positive constant $M_0$ with following property: given any $Y \geq 0$ there exist $T_0 = T_0(Y, \tau) \geq \tau$ such that, whenever $\|z_0\|_{H_0} \leq Y$ it follows that

$$\|U_g(t, \tau)z_0\|_{H_0} \leq M_0, \quad \forall t \geq T_0.$$

Consequently, the set

$$B_0 = \{z_0 \in H_0 : \|z_0\|_{H_0} \leq M_0\}$$

is a bounded uniformly (w.r.t $\sigma \in \Sigma$) absorbing set for $U_g(t, \tau)$ on $H_0$, that is, for any bounded (in $H_0$) subsets $B$, there is a $T_0 = T_0(\|z_0\|_{H_0}, \tau) \geq \tau$ such that

$$U_{g \in \Sigma} U_g(t, \tau)B \subset B_0$$

for every $t \geq T_0$.

Combining Lemma 3.1, we know that for any $\tau \in R$, $U_g$ maps the bounded set of $H_0$ into a bounded set of $H_0$ for all $t \geq \tau$, that is

**Corollary 3.3.** Given any $R > 0$, there is $M_R = M_R(R, \|z_0\|_{H_0})$ such that for all $\|z_0\|_{H_0} \leq R,$

$$\|U_g(t, \tau)z_0\|_{H_0} \leq M_R, \quad \forall t \geq \tau.$$

**Lemma 3.4.** Given any $R > 0$, let $z_{01}, z_{02} \in H_0$ be two initial data, and $\|z_{0i}\|_{H_0} \leq R(i = 1, 2)$. Then the following estimate holds,

$$\|U_g(t, \tau)z_{0i} - U_g(t, \tau)z_{0j}\|_{H_0} \leq Q(R)e^{\gamma(t-\tau)}(\|z_{0i} - z_{0j}\|_{H_0} + \|g - g\|_1).$$

(10)

for any $t \geq \tau$ and some $k = k(R)$.

C. Asymptotic Regularity

For the nonlinear function $f(u)$ from [2], we know that $f$ has the following decomposition

$$f = f_0 + f_1$$

where $f_0, f_1 \in C(R)$ and satisfy

$$f_0(s) \geq 0 \quad \text{for all} \quad s \in R,$$

$$|f_0(s)| \leq c(1 + |s|) \quad \text{for all} \quad s \in R,$$

$$|f_1(s)| \leq c(1 + |s|) \quad \text{for all} \quad s \in R \quad \text{with some} \quad \gamma < 1,$$

$$\lim_{|s| \to \infty} \frac{f_1(s)}{|s|} = -\lambda_1,$$

(14)

where $c, \lambda_1$ are positive constants and $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ with the Dirichlet boundary condition. Denote

$$\sigma = \min\{\frac{1}{4}, \frac{5 - \gamma}{2}\}.$$  

(15)

In order to obtain the regularity estimates later, we decompose the solution $U_g(t, \tau)z_0 = (u(t), v(t), \eta(t))$ into the sum:

$$U_g(t, \tau)z_0 = S(t, \tau)z_0 + K_g(t, \tau)z_0.$$ 

$S(t, \tau)z_0 = (v(t), \eta(t), \xi(t))$, $K_g(t, \tau)z_0 = (w(t), \eta(t), \xi(t))$ are the solutions the following equations respectively.
\[
\begin{align*}
\dot{v}_t - \Delta v_t - \Delta v - \frac{\mu(s) \Delta x^\xi(s)}{s} ds + f_g(v) &= 0, \\
\frac{d}{dt} \left( \int_0^t \int_0^t S(t, s) \Delta x^\xi(s) ds + f(u) - f_0(v) \right) &= g(x, t), \\
\left( v(t), v_t(t), x^\xi \right) &= z, \quad v|_{t=0} = 0, x^\xi|_{\partial\Omega} = 0, \\
\end{align*}
\]

(16)

and

\[
\begin{align*}
\dot{w}_t - \Delta w_t - \Delta w - \frac{\mu(s) \Delta x^\xi(s)}{s} ds + f(u) - f_0(v) &= g(x, t), \\
\frac{d}{dt} \left( \int_0^t \int_0^t S(t, s) \Delta x^\xi(s) ds + f(u) - f_0(v) \right) &= g(x, t), \\
\left( w(t), w_t(t), x^\xi \right) &= 0, \quad w|_{t=0} = 0, x^\xi|_{\partial\Omega} = 0.
\end{align*}
\]

(17)

We will establish a priori estimates about the solutions of (16) and (17), which are the basis of our works.

**Lemma 3.5.** For any initial data \( z_0 \in H_0 \) of the solutions of (16) satisfy the following estimates: There exists constant \( k_0 \) such that for every \( t \geq \tau \),

\[
\left\| S(t, \tau) z_0 \right\|_H = \left\| h(t) \right\|_H + \left\| v(t) \right\|_H + \left\| x^\xi \right\|_{\partial\Omega} \leq Q_t(\tau) \left\| z_0 \right\|_H e^{k_0(t-\tau)}
\]

where \( Q_t(\cdot) \) is an increasing function on \([0, \infty)\), \( Q_t \) and \( k_0 \) only depend on the \( H_0 \)-bound of \( z_0 \), but both are independent of \( \tau \).

**Proof.** Repeating word by word the proof of Theorem 3.2, that applies to the present case with \( S(t, \tau) z_0 \) in place of \( U_g(t, \tau) z_0 \) (with the further simplification that \( C = 0 \), for now \( f_1 = 0 \) and \( g = 0 \), it follows that

\[
\left\| S(t, \tau) z_0 \right\|_H = \left\| h(t) \right\|_H + \left\| v(t) \right\|_H + \left\| x^\xi \right\|_{\partial\Omega} \leq Q_t(\tau) \left\| z_0 \right\|_H e^{k_0(t-\tau)}.
\]

For the solution of (17), we have

**Lemma 3.6.** For any \( t \in R \), the solutions of (17) satisfy the following estimates: There exists constant \( k_0 \) such that for every \( t \geq \tau \),

\[
\left\| K_g(t, \tau) z_0 \right\|_{H_0} = \left\| h(t) \right\|_{H_0} + \left\| v(t) \right\|_{H_0} + \left\| x^\xi \right\|_{\partial\Omega} \leq Q_0(\tau) \left\| z_0 \right\|_{H_0} e^{k_0(t-\tau)},
\]

where \( Q_0(\cdot) \) is an increasing function on \([0, \infty)\), and \( \sigma \) is given in (15).

**Proof.** Multiplying (17) by \( A^* w_1(t) \), and integrating in \( dx \) over \( \Omega \), we get that

\[
\frac{d}{dt} \left( \int_0^t \int_0^t S(t, s) \Delta x^\xi(s) ds + f(u) - f_0(v) \right) = f(u) - f_0(v) + A^* w_1(t), \quad g(x, t).
\]

(18)

Similar to that in Theorem 3.2 above, we get

\[
-\int_0^t \int_0^t \mu(s) \Delta x^\xi(s) A^* x^\xi(s) ds dx = \frac{1}{2} \frac{d}{dt} \left\| x^\xi \right\|_{L^2}^2.
\]

(19)

and

\[
-\int_0^t \int_0^t \mu(s) \Delta x^\xi(s) A^* x^\xi(s) ds dx \geq \frac{\nu}{2} \left\| x^\xi \right\|_{L^2}^2.
\]

(20)

Next we deal with the nonlinearity, we have

\[
\left\| f(u) - f_0(v) \right\|_{L^2} \leq C \left\| h(t) \right\|_{H_0}^2 + \frac{1}{4} \left\| A^* w_1(t) \right\|_{L^2},
\]

(22)

Note that \( \sigma \leq \frac{3 - \gamma}{2} \), so we can get the following estimates

\[
\left\| f_1(v) \right\|_{L^2} \leq C + \frac{1}{4} \left\| w_1(t) \right\|_{L^2}.
\]

(23)

Moreover,

\[
\left\| A^* w_1(t) \right\|_{L^2} \leq C + \frac{1}{4} \left\| w_1(t) \right\|_{L^2}.
\]

(24)

Combined with (19)-(20) and (22)-(23), by (18), we have that

\[
\int_0^t \int_0^t S(t, s) \Delta x^\xi(s) ds + f(u) - f_0(v) \leq C \left\| g(t) \right\|_{H_0}^2 + C_0 \left\| h(t) \right\|_{H_0}^2.
\]

Applying the Gronwall’s inequality, we deduce that

\[
\left\| A^* w_1(t) \right\|_{L^2} \leq C \left\| g(t) \right\|_{H_0}^2 + C_0 \left\| h(t) \right\|_{H_0}^2.
\]
here \( k_i = C_{M_i} \) and \( C_{M_i} \) depend on \( \|z\|_{L_0} \).

**Lemma 3.7.** For any \( \varepsilon > 0 \) \( u(t) \) is decomposed as
\[
u(t) = v_i(t) + w_i(t),
\]
and \( v_i(t) \) satisfies: there is a positive constant \( M_1 = M_1(\|z\|_{L_0}) \) such that the following estimates are true
\[
\|v_i(t)\|_{L_1} \leq M_1,
\]
and
\[
\int_0^t \|v_i(t)\|_{L_0}^2 dt \leq \varepsilon (t-s) + C_{\varepsilon}, \quad \text{for all} \quad t \geq s \geq \tau. \tag{25}
\]

As well as \( w_i(t) \) satisfies the following estimate
\[
\|w_i(t)\|_{L_1} \leq K_\varepsilon \quad \text{for all} \quad t \geq \tau, \tag{26}
\]
with the constants \( C_{\varepsilon} \) and \( K_\varepsilon \) depending on \( \varepsilon, \|z\|_{L_0} \) and \( \|s\|_{L_0} \), but both being independent of \( \tau \).

The proof of this lemma is similar to that in Sun [14].

In what follows we begin to establish the asymptotic regularity of the solutions of (1).

**Lemma 3.8.** There exists constant \( Y_0 \) which depends only on the \( H_0 \)-bounds of \( B(\subset H_0) \), such that for any \( \tau \in R \)
\[
\|K_\varepsilon(t,\tau)z\|_{H_\varepsilon} \leq Y_0, \quad \text{for all} \quad t \geq \tau \quad \text{and} \quad z \in B,
\]
where \( \sigma \) is given in (15).

**Proof.** Taking inner product of the first equation of (17) and \( \Delta^+ (w_i + \varepsilon w) \) (\( \varepsilon \) is an positive undetermined constant), we get that
\[
\left\{ w_i - \Delta w - \Delta w_i - \int_0^t \mu(s) \Delta^+ (s) ds, \Delta^+ (w_i + \varepsilon w) \right\} = -\left\{ f(u) - f_i(v), \Delta^+ (w_i + \varepsilon w) \right\} + \left\{ g(x,t), \Delta^+ (w_i + \varepsilon w) \right\}, \tag{27}
\]
In the following, we will deal with the left side of (27) one by one. Similar to that (19) and (20), we get that
\[
-\left\{ \int_0^t \mu(s) \Delta^+ (s) ds, \Delta^+ (w_i + \varepsilon w) \right\} \geq \frac{1}{2} \left\| \int_0^t \mu(s) \Delta^+ (s) ds \right\|_{L_\infty} \left\| \Delta^+ (w_i + \varepsilon w) \right\|_{L_1}.
\]
Now we rewrite (27) as
\[
\frac{d}{dt} E_2(t) + I_2(t) = -\left\{ f(u) - f_i(v), \Delta^+ (w_i + \varepsilon w) \right\} + \left\{ g(x,t), \Delta^+ (w_i + \varepsilon w) \right\} \tag{28}
\]
here
\[
E_2(t) = \frac{1}{2} \left\| \frac{\varepsilon^2}{2} w_i \right\|_2^2 + \varepsilon \left\langle w, A^w w \right\rangle + \frac{1 + \varepsilon}{2} \left\| w \right\|_2^2 + \frac{1}{2} \left\langle A w, A^w w \right\rangle \tag{29}
\]
and
\[
I_2(t) = -\varepsilon \left\| \frac{\varepsilon^2}{2} w_i \right\|_2^2 + \varepsilon \left\| w \right\|_2^2 + \varepsilon \left\| w_i \right\|_2^2 - \varepsilon \left\| \frac{\varepsilon^2}{2} w \right\|_2^2 \tag{30}
\]
Applying the Hölder’s inequality in (29), we get that
\[
E_2(t) \leq \alpha_1 \left\| w \right\|_2^2 + \left\| \left\| w \right\|_2^2 + \varepsilon \left\| w \right\|_2^2 \tag{31}
\]
where \( \alpha_1 = \max \left\{ \frac{1 + \varepsilon}{2}, \frac{1}{\lambda_1}, \frac{\varepsilon}{2} \right\} \).

On the other hand, we have
\[
E_2(t) \geq \frac{1}{2} \left( 1 - \varepsilon^2 \right) \left\| w \right\|_2^2 + \frac{1}{2} \left( 1 - \varepsilon \right) \left\| w \right\|_2^2 + \frac{1}{2} \left\| \frac{\varepsilon^2}{2} w \right\|_2^2, \tag{32}
\]
choose
\[
\varepsilon \leq \frac{1}{2} \min \left\{ 1, \sqrt{\lambda_1} \right\}. \tag{33}
\]
Let \( \beta_1 = \min \left\{ \frac{1}{2} (1 - \varepsilon), \frac{1}{2} (1 - \varepsilon^2) \right\} > 0 \), then
\[
E_2(t) \geq \beta_1 \left\| w \right\|_2^2 + \left\| \left\| w \right\|_2^2 + \varepsilon \left\| w \right\|_2^2 \tag{34}
\]
Toward \( I_2(t) \), we have
\[
I_2(t) \geq \frac{\varepsilon}{2} \left( 1 - 1 - \frac{\varepsilon}{2} \right) \left\| w \right\|_2^2 + \frac{1}{2} (\delta - \varepsilon) \left\| \frac{\varepsilon^2}{2} w \right\|_2^2 \tag{35}
\]
Combined with (31), choose
\[
\varepsilon = \frac{1}{2} \min \left\{ \frac{\lambda_1}{1 + \lambda_1}, \delta, \sqrt{\lambda_1} \right\}.
\]
Let \( \alpha_2 = \frac{1}{2} \min \left\{ \varepsilon, 2(1 - (1 + \varepsilon) \varepsilon), \delta - \varepsilon \right\} \).

\[
I_2 \geq \alpha_2 \left\| w \right\|_2^2 + \left\| \left\| w \right\|_2^2 + \varepsilon \left\| w \right\|_2^2 \tag{36}
\]
From Corollary 3.3 and Lemma 3.5, there is a positive constant $M_2 = M_2(\|z\|_{L^2})$ such that

$$\|K_0^\tau(t, \tau) z\|_{L^2} \leq M_2$$

holds for any $\tau \in R$.

Since $1 + \delta < 1$, employing the interpolation inequality, we can get that

$$\left\|g(t), A^\tau(w_1 + \varepsilon w)\right\| \leq C \left\|g(t)\right\|_{L^\infty} + \frac{\alpha_2^2}{8} \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2,$$  \hspace{1cm}(37)

and employing Lemma 3.7 to deal with the nonlinear term:

$$\left\|f(u) - f(v), A^\tau(w_1 + \varepsilon w)\right\| \leq \left\|f(u) - f(v), A^\tau(w_1 + \varepsilon w)\right\| + \left\|f(v), A^\tau(w_1 + \varepsilon w)\right\|.$$  \hspace{1cm}(38)

From (7) and Lemma 3.5, we have

$$\left\|f(u) - f(v), A^\tau(w_1 + \varepsilon w)\right\| \leq CM_2 + C \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2.$$  \hspace{1cm}(39)

Using Lemma 3.7, we have

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\|.$$  \hspace{1cm}(40)

and

$$\int_0^1 |v(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq M_1 \left\|v(\tau)\right\|_{L^2} \left\|A^\tau(w_1 + \varepsilon w)\right\|.$$  \hspace{1cm}(41)

Therefore, note that $\sigma = \min\{1, 5 - \gamma\}$, we have

$$\frac{12}{5} < \frac{6}{1 + 6\sigma} < 6,$$  \hspace{1cm}(42)

where $K_2$ is given in (26).

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{2K_2^2M_2^2}{\alpha_2} \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2,$$  \hspace{1cm}(43)

where $Q_1(\|z\|_{L^2})$ from Lemma 3.5.

Substitute (40)-(43) into (39), we get that

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{C\left\|v(\tau)\right\|_{L^2}^2 + \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2}{K_2} + K_2.$$  \hspace{1cm}(44)

where $K_1 = CM_2 + \frac{2\alpha_2^2M_2^2}{\alpha_2}$.

Similarly,

$$\int_0^1 |v(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq K_2 + \frac{\alpha_2^2}{8} \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2,$$  \hspace{1cm}(45)

Moreover, it follows Lemma 3.7,

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{Q_1(\|z\|_{L^2})}{K_2},$$  \hspace{1cm}(46)

Then Gronwall’s inequality yields, for any $0 > 0$

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{Q_1(\|z\|_{L^2})}{K_2},$$  \hspace{1cm}(47)

Hence, combining the above estimates into (28), we see that for all $t \geq \tau$,

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq K_2 + K_2.$$  \hspace{1cm}(46)

Then Gronwall’s inequality yields, for any $t \geq T > \tau$

$$E_2(t) \leq \beta E_2(T)e^{-\beta(t-T)} + \rho,$$

here $\beta > 0$ is a constant which depended on initial data and $\gamma, \rho$ are positive constants which depended on initial data.

At the last, by Lemma 3.6, (31), (34) and noting that $T > \tau$ is fixed, then the proof is completed.

Lemma 3.9. Assume $B_\sigma$ is bounded in $H_\sigma$. Then there exists a constant $M_\sigma(> 0)$ which only depends on the $H_\sigma$-bounds of $B_\sigma$ such that for any $t \geq \tau$

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq M_\sigma \text{ for all } t \geq \tau \text{ and } z \in B_\sigma.$$

Proof Multiply (1) by $A^\tau(u_1 + \varepsilon u)$ ( $\varepsilon$ is a positive undetermined constant), we get that

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq M_\sigma \text{ for all } t \geq \tau \text{ and } z \in B_\sigma.$$

Similarly,

$$\int_0^1 |v(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq K_2 + \frac{\alpha_2^2}{8} \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2,$$  \hspace{1cm}(45)

Moreover, it follows Lemma 3.7.

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{Q_1(\|z\|_{L^2})}{K_2},$$  \hspace{1cm}(46)

Then Gronwall’s inequality yields, for any $t \geq T > \tau$

$$E_2(t) \leq \beta E_2(T)e^{-\beta(t-T)} + \rho,$$

here $\beta > 0$ is a constant which depended on initial data and $\gamma, \rho$ are positive constants which depended on initial data.

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$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq M_\sigma \text{ for all } t \geq \tau \text{ and } z \in B_\sigma.$$

Similarly,

$$\int_0^1 |v(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq K_2 + \frac{\alpha_2^2}{8} \left\|w_1\right\|_{L^2}^2 + \left\|w\right\|_{L^2}^2,$$  \hspace{1cm}(45)

Moreover, it follows Lemma 3.7.

$$\int_0^1 |w(\tau)|^2 \|A^\tau(w_1 + \varepsilon w)\| \leq \frac{Q_1(\|z\|_{L^2})}{K_2},$$  \hspace{1cm}(46)

Then Gronwall’s inequality yields, for any $t \geq T > \tau$

$$E_2(t) \leq \beta E_2(T)e^{-\beta(t-T)} + \rho,$$

here $\beta > 0$ is a constant which depended on initial data and $\gamma, \rho$ are positive constants which depended on initial data.
\[ E_i(t) = \frac{1}{2} \| \frac{\sigma}{2} u \|_2^2 + \epsilon (u, A^\tau u) + \frac{1 + \delta}{2} \| \sigma \|_{L^\sigma}^2 + \frac{1}{2} \| \sigma \|_2^2 - \epsilon \| \sigma \|_{L^\sigma}^2 + \epsilon \langle A u, A^\tau u \rangle, \]  

and

\[ I_j(t) = -\epsilon \| \frac{\sigma}{2} u \|_2^2 + \epsilon \| \sigma \|_{L^\sigma}^2 + \frac{1}{2} \| \sigma \|_2^2 - \epsilon \| \sigma \|_{L^\sigma}^2 + \frac{\delta}{2} \| \sigma \|_{L^\sigma}^2 - \epsilon \| \sigma \|_{L^\sigma}^2 \| \sigma \|_2. \]

Applying the Hölder’s inequality in (48), we get that

\[ E_i(t) \leq \alpha_i (\| u \|_2^2 + \| \sigma \|_{L^\sigma}^2 + \| \sigma \|_2^2), \]  

where \( \alpha_i \) from (31).

On the other hand, we have

\[ E_i(t) \geq \beta_i (\| u \|_2^2 + \| \sigma \|_{L^\sigma}^2 + \| \sigma \|_2^2), \]

where \( \beta_i = \min \{ \frac{1}{2} (1 - \epsilon), \frac{1}{2} (1 - \epsilon^2) \} > 0 \) (from (34)).

Toward \( I_j(t) \), we have

\[ I_j(t) \geq \frac{\delta}{2} \| \sigma \|_{L^\sigma}^2 - \epsilon \| \sigma \|_{L^\sigma}^2 + \frac{1}{2} \| \sigma \|_2^2 - \epsilon \| \sigma \|_{L^\sigma}^2 \]

Combined with (52), choose

\[ \epsilon \leq \frac{1}{2} \min \{ \frac{\lambda_i}{1 + \lambda_i}, \delta, \sqrt{\lambda_i} \}. \]

Similar to (36), let \( \alpha_2 = \frac{1}{2} \min \{ \epsilon, 2(1 - 1 + \frac{1}{\lambda_i}) \epsilon, \delta - \epsilon \}. \)

Using Lemma 3.1 and integrating over \([ \tau, T] \), we get that

\[ E_i(t) \leq \alpha_i \| u \|_2^2 e^{-\omega(t-\tau)\omega} + \frac{m_\tau}{1 - e^{-\omega}}. \]
where
\[ w = \frac{\alpha_2}{4\epsilon}, \quad m_1 = C_u \text{ (from (25)),} \quad m_2 = C_2 + C_3 + \| e \|_2. \]

We then complete the proof.

**Lemma 3.10.** For each \( \theta \in [\sigma, 1] \), let \( B \) be any bounded subset of \( H_\theta \). Then there exists a constant \( M_B \) which only depends on the \( H_\theta \)-bounds of \( B \), such that for any \( \tau \in R \),
\[ \| U_\theta (t, \tau) z_\tau \|_{B_{M_B}} \leq M_B \quad \text{for all} \quad t \geq \tau \quad \text{and} \quad z_\tau \in B. \]

**Lemma 3.11.** For each \( \theta \in [\sigma, 1 - \sigma] \), if the initial data set \( B \) be any bounded subset of \( H_\theta \), then the decomposed ingredient \((w(t), w_\tau(t))\) (the solutions of (17)) satisfies, for any \( \tau \in R \),
\[ \| K_\theta (t, \tau) z \|_{B_{Y_\theta}} \leq Y_\theta \quad \text{for all} \quad t \geq \tau \quad \text{and} \quad z \in B, \]
where the constant \( Y_\theta \) only depends on the \( H_\theta \)-bounds of \( B \).

**Theorem 3.12.** There exist a bounded (in \( H_\theta \)) set \( B_0 \subset H_\theta \), a positive constants \( \epsilon \) and a monotonically increasing function \( Q(\cdot) \) such that, for any \( \theta \in R \), any \( g \in \sum \), \( \tau \in R \) and \( t \geq \tau \), the following estimate holds:
\[ \text{dist}_{H_\theta} (U_{\theta+\tau} (t, \tau) B_0, B_0) \leq Q(\| B_0 \|_{H_\theta}) e^{Q(t-\tau)}. \]

where \( \text{dist}_{H_\theta} (\cdot, \cdot) \) denotes the usual Hausdorff semi-distance in \( H_\theta \).

**Proof** Let \( B_0 \) be the bounded uniformly (w.r.t \( \sigma \in \sum \)) absorbing set in \( H_\theta \) (see Theorem 3.2).

By Lemma 3.5 and Lemma 3.8, set \( A_\sigma = \{ z \in H_\theta : \| z \|_\sigma \leq Y_\theta \} \) then
\[ \text{dist}_{H_\theta} (U_{\theta+\tau} (t, \tau) B_0, A_\sigma) \leq \text{dist}_{H_\theta} (S(\tau, \tau) B_0, A_\sigma) \leq Q(\| B_0 \|_{H_\theta}) e^{Q(t-\tau)}, \]
where \( Y_\theta \) is a constant from Lemma 3.8 corresponding to \( B_0 \).

Using \( A_\sigma \) to replace \( B_0 \) in Lemma 3.11 and Lemma 3.5, then there is \( A_{\sigma+\tau} \subset H_\theta \) which is bounded in \( H_\theta \) such that
\[ \text{dist}_{H_\theta} (U_{\theta+\tau} (t, \tau) A_{\sigma+\tau}, A_{\sigma+\tau}) \leq \text{dist}_{H_\theta} (S(\tau, \tau) A_{\sigma+\tau}, A_{\sigma+\tau}) \leq Q(\| A_{\sigma+\tau} \|_{H_\theta}) e^{Q(t-\tau)} \]
for two appropriate constants \( C \) and \( k_\sigma \).

Since \( \sigma = \min \left\{ \frac{1}{4}, \frac{5-\gamma}{2} \right\} \) is fixed, by finite steps, we can infer that there is a bounded (not only in \( H_\theta \), but also \( H_\tau \)) set \( B_1 \subset H_\tau \) such that
\[ \text{dist}_{H_\tau} (U_{\tau+\tau} (t, \tau) B_1, B_1) \leq Q(\| B_1 \|_{H_\tau}) e^{Q(t-\tau)}. \]

Note further that all the constants in (62) only depend on \( \| B_0 \|_{H_\theta} \) and \( \| e \|_2 \).

Now, for any bounded (in \( H_\theta \)) \( B \), from Theorem 3.2 There is \( T_0 \geq \tau \) such that
\[ \bigcup_{\sigma \geq T_0} U_{\theta+\tau} (t, \tau) B \subset B_0 \quad \text{for all} \quad t \geq T_0. \]

Combined with Lemma 3.4, it follows that
\[ \text{dist}_{H_\theta} (U_{\tau+\tau} (t, \tau) B, B_0) \leq Q e^{Q(t-\tau)} e^{Q(t-\tau)}, \]
where \( Q = \sup \{ \| U_{\tau+\tau} (t, \tau) B \|_{H_\theta} : g \in \sum, \tau \leq t \leq T_0 \} < \infty. \)

Finally, we apply Lemma 2.1, again to (62) and (63), and the proof of Theorem is completed.

**IV. UNIFORMLY ATTRACTORS**

Now collecting Theorem 3.2, Lemma 3.5, and Theorem 3.12, we establish that \( \{ U_{\theta+\tau} (t, \tau) \}, g \in \Sigma \) corresponding to (1) is asymptotically compactness. Therefore, by means of well-known results of the theory of dynamical systems we get that the family of processes \( \{ U_{\theta+\tau} (t, \tau) \}, g \in \Sigma \) corresponding to (1), posses a compact (in \( H_\theta \)) uniform (w.r.t \( g \in \Sigma \)) attractor \( \mathcal{A} \), and \( \mathcal{A} \subset H_\theta \). We remark that the above existence does not require any continuity of the family of processes. However, in order to obtain the explicit form of \( \mathcal{A} \), we need some continuity. Moreover, since the symbol space \( \Sigma \) now has only weak compactness, we need to verify the corresponding of weak continuity. First, by the results of Chepyzhov and Vishik[6], we see that \( \Sigma \) with the local weak convergence topology of \( L_{\text{weak}}^1 (R; L^2 (\Omega)) \) forms a sequentially compact and metrizable complete space. We denote the equivalent metric by \( d(\cdot, \cdot) \). Thus \( (\Sigma, d) \) is a compact metric space. Moreover, through Lemma 4.1, Chapter V[6], we also have the following conclusion.

**Lemma 4.1.** [13] The translation semigroup \( \{ T(t) \}_{t \geq 0} \) acts on \( \sum \) (i.e., \( T(t) g(x, s) = g(x, t + s) \) for any \( g \in \sum \) and any \( t \geq 0 \) is invariant and continuous in \( \Sigma \) with respect to the local weak convergence topology of \( L_{\text{weak}}^1 (R; L^2 (\Omega)) \), equivalently, with respect to the metric \( d \).

In the following, we also recall an useful lemma, whose proof is simple and we omit it.
Lemma 4.2. [13] Let \( X \) be a reflexive Banach and \( x_n \xrightarrow{N} 0 \) in \( X \), then for each compact (in \( X^* \)) subset \( B \subset X^* \), the uniform convergence hold: For any \( \varepsilon > 0 \) there is a \( N_\varepsilon \), depending only on \( \varepsilon \), such that
\[
|f(x_n - x)| \leq \varepsilon \quad \text{for all} \quad n \geq N_\varepsilon \quad \text{and all} \quad f \in B \quad (64)
\]

Theorem 4.3. The family of processes \( \{U_g(t, \tau)\}, \sigma \in \Sigma \), corresponding to (1.1), has a compact uniform (w.r.t. \( \sigma \in \Sigma \)) attractor \( \mathfrak{A} \) in \( H_0 \). Moreover, this attractor is bounded in \( H_1 \) and can be decomposed as follows
\[
\mathfrak{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0)
\]
where \( \mathcal{K}_\sigma \) is the kernel of the process \( U_g \), and \( \mathcal{K}_\sigma(0) \) is the kernel section at time \( \Omega \).

Proof. We only need to verify continuity claim on the attractor \( \mathfrak{A} \) in \( H_0 \), i.e., for the attractor \( \mathfrak{A} \) in \( H_0 \), \( \mathfrak{A} \subset H_0 \) any fixed \( \tau \in \mathbb{R} \) and \( t \geq \tau \), if \( z_{\tau \omega} \to z_{\tau} \) in \( \mathfrak{A} \) and \( g_{\tau \omega} \to g_{\tau} \) with respect to the local weak convergence topology of \( L_w^2(R;L^2(\Omega)) \), then \( U_{g_{\tau \omega}}(t, \tau)z_{\tau \omega} \) converges to \( U_{g_{\tau}}(t, \tau)z_{\tau} \) in \( \mathfrak{A} \).

Denoted
\[
z_{\tau} = z_{\tau \omega} - z_{\tau}, \quad (u(t),u(t),p') = U_{g_{\tau}}(t, \tau)z_{\tau}, \quad (i = 1, 2)
\]
and
\[
(w(t),w(t),\zeta(t)) = U_{g_{\tau}}(t, \tau)z_{\tau} - U_{g_{\tau}}(t, \tau)z_{\tau}, \quad z_{\tau}, \quad \zeta(t) = z_{\tau}, \quad w|_{\tau \omega} = 0.
\]

Since \( \mathfrak{A} \) is bound in \( H_1 \), following Theorem 3.12, then there is a positive constant \( R_0 \) such that
\[
\sup_{\sigma \in \Sigma} \sup_{\tau \omega \leq \tau} \sup_{t \geq \tau} \|U_{g_{\tau \omega}}(t, \tau)\mathfrak{A}|_{H_0} \leq R_0 < \infty \quad (67)
\]

Multiplying (66) by \( w(t) \) and using (67), it follows that
\[
\frac{d}{dt} \left[ \|w(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^{\infty}}^2 \right] + 2 \|w(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^{\infty}}^2 \leq G(t) - G_{\tau}(t), \quad w(t) > 0
\]
and by integrating over \([\tau, t]\), then we get, for each \( \tau \leq t \leq T \),
\[
\left| \|w(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^{\infty}}^2 \right| + \int_{\tau}^T \left( G(t) - G_{\tau}(t) \right) \, ds \leq 0.
\]

By Theorem 3.2, then we have
\[
\mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0}) \leq \mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0})
\]
and
\[
\mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0}) \leq \mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0})
\]
then
\[
\mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0}) \leq \mathbb{P}(\mathfrak{B}(\tau, T) \in \mathfrak{A}|_{H_0})
\]
uniformly on a compact subset of \( L^2(t, T; L^2(\Omega)) \).

Based on the continuity claim above, and by constructing a skew-product flow on \( \mathfrak{A} \times \Sigma \) and applying Theorem 3.1, IV[6], then the structure equality (65) is proved. So the proof is completed.

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