

Periodic solutions to ZK-MEW equation in the form of Lamé equation

Chun-Huan Xiang^{1,a}, Hong-Lei Wang^{2,b}

¹School of Public Health and Management, Chongqing Medical University, Chongqing, 400016, P.R. China

²College of medical informatics, Chongqing Medical University, Chongqing, 400016, P. R. China

^aemail: w825900@163.com, ^bemail: w8259300@126.com

Abstract: The periodic solutions to ZK-MEW equation is investigated based on an auxiliary Lamé equation and the perturbation method. The periodic solutions to ZK-MEW equation is given in the form of Jacobi functions and expressed by the hyperbolic functions, the trigonometric functions with different modulus m . The results are simply discussed. This method is more powerful to seek the exact solutions of the nonlinear partial differential equations in mathematical physics.

Keywords: nonlinear, ZK-MEW equation, Lamé equation, evolution equation

1. Introduction

The investigation about exact solutions of nonlinear equations is an important subject because they play important role in understanding the nonlinear problems. Recently, many authors presented various powerful method to deal with this interest subject, such as Backlund transformation [1], Darboux transformation [2], the extended tanh-function method [3], the F-expansion method [4], projective Riccati equations method[5], the Jacobian elliptic functions expansion method[6] and so on [7-10]. Recently, vast research results [11-15] have been obtained in this field. The aim of this paper is to extend the Lamé equation method to solve the nonlinear differential equations ZK-MEW equation [16].

2. The nonlinear ZK-MEW equation and Lamé equation method

The ZK-MEW equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [17]. The nonlinear ZK-MEW equation is given in the form as follow

$$(u_t + a(u^3)_x + (bu_{xt} + ru_{yy}))_x = 0 \quad (1)$$

where a , b and r are known constants. Using the wave variable $\xi = x + y - ct$ and proceeding as before we find

$$cu' + a(u^3)' - bcu''' + ru''' = 0 \quad (2)$$

Integrating (2) with respect to ξ and neglecting constants of integration we obtain

$$-cu + au^3 + (r - bc)u'' = 0 \quad (3)$$

Lamé equation with a kernel function expressed in terms of $dn(\xi)$ reads

$$\frac{d^2 \Lambda_n^{(p)}(\xi)}{d\xi^2} + [p(p+1)dn^2(\xi) - B_n^{(p)}] \Lambda_n^{(p)}(\xi) = 0, \quad p = 1, 2, \dots, \quad n = 1, 2, \dots, 2p+1. \quad (4)$$

Here $\Lambda_n^{(p)}(\xi)$ is the n th eigenfunction of Lamé Eq. (4), corresponding to the eigenvalue $B_n^{(p)}(\xi)$

For Jacobi elliptic functions, we have

$$cn^2 \xi = 1 - sn^2 \xi, \quad dn^2 \xi = 1 - m^2 sn^2 \xi, \text{ and } \frac{d}{d\xi} sn \xi = cn \xi dn \xi, \quad \frac{d}{d\xi} cn \xi = -sn \xi dn \xi, \quad \frac{d}{d\xi} dn \xi = -m^2 sn \xi cn \xi. \quad (5)$$

$sn(\xi)$, $cn(\xi)$, $dn(\xi)$ are Jacobi elliptic sine function, cosine function and Jacobi elliptic function of the third kind with the modulus m ($0 < m < 1$), respectively. When $m \rightarrow 0$ or $m \rightarrow 1$, Jacobi elliptic functions asymptotically go into trigonometric or hyperbolic one:

$$\begin{aligned} \operatorname{sn} \xi &\rightarrow \sin \xi, \quad \operatorname{cn} \xi \rightarrow \cos \xi, \quad \operatorname{dn} \xi \rightarrow 1, \quad \text{when } m \rightarrow 0; \\ \operatorname{sn} \xi &\rightarrow \tanh \xi, \quad \operatorname{cn} \xi \rightarrow \operatorname{sech} \xi, \quad \operatorname{dn} \xi \rightarrow \operatorname{sech} \xi, \quad \text{when } m \rightarrow 1. \end{aligned} \quad (6)$$

For a given nonlinear equation as the follow

$$G(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0$$

Where $u_x = \partial u / \partial x$, $u_{xx} = \partial^2 u / \partial x^2$, $u_{xt} = \partial^2 u / \partial x \partial t$, ... with the expression as $\xi = x + y - ct$, where c denotes the wave speed, then leads to an ordinary differential equation.

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots) = 0 \quad (7)$$

The minimal expand is used to obtain the approximation evolution solution for nonlinear equation.

$$u(\xi) = u_0(\xi) + pu_1(\xi) + p^2u_2(\xi) + \dots \quad (8)$$

Where p is small parameter, $u_0(\xi)$, $u_1(\xi)$, $u_2(\xi)$ represent the zeroth-order, first-order and second-order solution, respectively.

Substituting Eq. (8) into Eq.(7) and equating to zero the coefficients of all powers of p yields a set of the second-order differential equations in terms of $u_0(\xi)$, $u_1(\xi)$, $u_2(\xi)$, which called the zeroth-order, first-order, second-order, ... equations, respectively. All equations can be solved by using auxiliary Lamé equation method.

3. Numerical example

We employ the Lamé equation method for the nonlinear equations (3), the useful expression (8) is used. Substituting Eq. (8) into Eq. (3), and equating the coefficients of all powers of p to zero, the first three equations as follows:

$$au_0^3 - cu_0 - bcu_0'' + ru_0'' = 0 \quad (9)$$

$$3au_0^2u_1 - cu_1 - bcu_1'' + ru_1'' = 0 \quad (10)$$

$$3au_0u_1^2 + 3au_0^2u_2 - cu_2 - bcu_2'' + ru_2'' = 0 \quad (11)$$

Eq.(9) can be solved by using auxiliary elliptic equation method and suppose.

$$u_0 = a_0 + a_1 \operatorname{dn}(\xi) + a_2 \operatorname{dn}(\xi)^2 \quad (12)$$

Submitting the equation (12) into Eq.(9), the parameter a_0 , a_1 , a_2 can be easily determined

$$u_0 = \frac{(bc - r)(m^2 - 2)}{c} \operatorname{dn}(\xi) \quad (13)$$

Putting Eq. (13) into Eq.(10) yields

$$u_1'' + \left(\frac{3a(r - bc)(m^2 - 2)^2}{c^2} \operatorname{dn}(\xi)^2 - c \right) u_1 = 0, \quad (14)$$

By employing Eq. (5) and (6), we can obtain the solution as the follow

$$u_1 = B \operatorname{sn}(\xi) \operatorname{cn}(\xi) \operatorname{dn}(\xi) \quad (15)$$

Submitting the equation (13) and (15) into Eq. (11), we obtain

$$\frac{3a(2 - m^2) \operatorname{dn}(\xi)}{c} (B \operatorname{sn}(\xi) \operatorname{cn}(\xi) \operatorname{dn}(\xi))^2 + \left(\frac{3a(2 - m^2)^2 \operatorname{dn}(\xi)^2}{c^2} - \frac{c}{r - bc} \right) u_2 + u_2'' = 0 \quad (16)$$

The solution of Eq.(16) is supposed as

$$u_2 = n_0 + n_1 \operatorname{dn}(\xi)^2 + n_2 \operatorname{dn}(\xi)^4 \quad (17)$$

Submitting the equation (17) into Eq.(16) and equating the coefficient of $\operatorname{dn}(\xi)$ to zero, we obtain the following equation:

$$-2n_1 + 2m^2n_1 - \frac{cn_0}{r - bc} = 0$$

$$\frac{12a - 12am^2 + 3am^4}{c^2}n_0 + (8 - 4m^2 - \frac{c}{r-bc})n_1 + 12(m^2 - 1)n_2 = 0$$

$$\frac{12a - 12am^2 + 3am^4 - 6c^2}{c^2}n_1 + (32 - 16m^2 - \frac{c}{r-bc})n_2 = 0$$

From the above equations, we obtain the solution as

$$n_1 = \frac{cn_0}{2(r-bc)(m^2-1)}$$

$$n_2 = \frac{2(12a - 12am^2 + 3am^4)(r-bc)^2(m^2-1) + ((8-4m^2)(r-bc)-c)c^3}{-24c^2(r-bc)^2(m^2-1)^2}n_0$$

where n_0 is a nonzero constant parameter.

The Eq. (17) is read as

$$u_2 = n_0 + \frac{cn_0}{2(r-bc)(m^2-1)}dn(\xi)^2 + \frac{2(12a - 12am^2 + 3am^4)(r-bc)^2(m^2-1) + ((8-4m^2)(r-bc)-c)c^3}{-24c^2(r-bc)^2(m^2-1)^2}n_0dn(\xi)^4 \quad (18)$$

Submitting Eq. (13) (15) and (18) into Eq.(8), we obtain the evolution solutions for Eq.(3) as the follow

$$u(\xi) = \frac{(bc-r)(m^2-2)}{c}dn(\xi) + pBsn(\xi)cn(\xi)dn(\xi) + n_0 + \frac{cn_0}{2(r-bc)(m^2-1)}dn(\xi)^2 +$$

$$\frac{2(12a - 12am^2 + 3am^4)(r-bc)^2(m^2-1) + ((8-4m^2)(r-bc)-c)c^3}{-24c^2(r-bc)^2(m^2-1)^2}n_0dn(\xi)$$

The solution for nonlinear ZK-MEW equation in the form Eq.(1) is written as

$$u(x, y, t) = \frac{(bc-r)(m^2-2)}{c}dn(x+y-ct, m) + pBsn(x+y-ct, m)cn(x+y-ct, m)dn(x+y-ct, m)$$

$$+ \frac{2(12a - 12am^2 + 3am^4)(r-bc)^2(m^2-1) + ((8-4m^2)(r-bc)-c)c^3}{-24c^2(r-bc)^2(m^2-1)^2}n_0dn(x+y-ct, m)^4 \quad (19)$$

$$+ n_0 + \frac{cn_0}{2(r-bc)(m^2-1)}dn(x+y-ct, m)^2$$

When $m \rightarrow 0$ or $m \rightarrow 1$, the Eq. (19) is reduced into trigonometric or hyperbolic solutions for Eq. (3)

The simulation figure for Eq.(19) is shown in Fig. 1, the parameters are $b=0.1$; $c=5$; $m=0.2$; $r=0.25$; $p=0.001$; $B=2$; $n_0 = 0.4$; $t=0.1$; $a=0.12$; $x \in (0,5)$, $y \in (0,5)$, respectively.

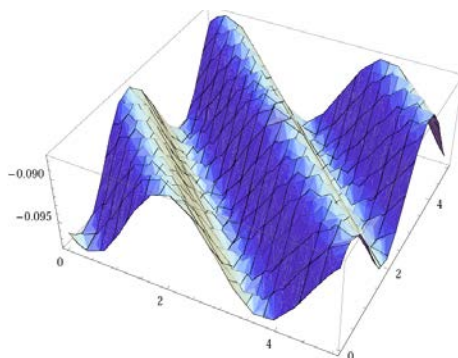


Fig.1 The simulation result is shown for Eq. (19)

4. Conclusions

The nonlinear ZK-MEW equation is investigated in this manuscript. The Lamé equation method was successfully used to establish travelling wave solutions, which performance is reliable and effective.

Many well known nonlinear wave equations were handled by this method. The simulation figures for the evolution solutions are shown. We believe that this method should play an important role for finding exact solutions in the mathematical physics.

5. References

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