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Network Data Envelopment Analysis with Intermediate Products Structure

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Abstract. Traditional Data envelopment analysis models make no hypothesis concerning the internal operations in a "black box". To open the "black box", we put forward a type of network DEA structure with several sub-DMUs linked by intermediate products. The input node and output node also represent the entrance and exit of the box. However, the input is assignable in the paper. We suggested a suitable model to deal with intermediate products between divisions with assignable inputs in the network structure. The model enables us to pry into the internal operations of DEA by network structure with intermediate products and assignable input. We finally illustrate the calculation procedure of the proposed algorithm by a numerical example.

1. Introduction

Data envelopment analysis (DEA), originated by Charnes, Cooper, and Rhodes, is a non-parametric algorithm for assessing the relative efficiency of homogeneous decision making units (DMUs), which use similar inputs to produce similar outputs [1]. DEA measures the efficiency of each DMU relative to an estimated production possibility frontier constructed by all DMUs. Conventional DEA approach is based on the idea of "black box", which requires no assumption on the appearance of the frontier surface as well as it makes no hypothesis concerning the internal operations of a DMU. However, the practical transformation process is normally not modeled explicitly. We can often meet actual problem with some kinds of network structure.

Network DEA allows us to look into these "black boxes" and to measure organizational performance and its component performance (see, e.g., [2-5]). The development of network DEA model has come through four stages: The first, Färe et al. [6] researched allocation of farmland to various crops, allows for allocation of a (fixed) factor or input among alternative uses. This general network structure could also be utilized to introduce allocation of a budget or allocation of resources across units or branches. This is the rudiment of primary network DEA model. The second, Mickael et al. [7] put forward a type of network model that allows inclusion of customer satisfaction in efficiency and productivity measures. The network framework consists of a production node and a consumption node, offering flexibility in modeling the production and consumption process where a firm-specific allocation of input resources to production and customer oriented activities is allowed. In this stage, intermediate products were emphasized. The third, Färe et al. [8] set up dynamic DEA model which considered time factor. The dynamic DEA model suggested here is applied to study the dynamic efficiency of APEC (Asian-Pacific Economic Community) countries. The network formulation is a dynamic DEA model in which some outputs at period t are inputs in the next period, t+1. However, the third stage neglects the structure of network itself which discussed in the second phase. Therefore, it is a simple network model with time parameter, but not real dynamic network DEA model. Fourthly, Kaoru Tone et al. [9] handle intermediate products formally to evaluate divisional efficiencies along with the whole efficiency of DMUs by a slack-based measure (SBM). They deals with any network structure in which nodes were connected and intermediates were directed between each other.

The rest of this paper unfolds as following. In Section 2, we introduce the conventional DEA model. In Section 3, we formulate network DEA model with intermediates products and assignable

inputs. In Section 4, we exhibit an illustrative example to explain the operational process of the network model. Some conclusions are presented in Section 5.

2. Conventional DEA Model

DEA is a widely used mathematical programming approach for comparing the multiple inputs and outputs of a set of homogenous DMUs by evaluating their relative efficiency. Suppose that there are n DMUs to be measured where each DMU_j, j = 1, 2, ... , n, consumes m inputs, x_{ij} (i = 1, 2, … , m) and producing s outputs, y_{rj} (r = 1, 2, … , s), which can be described as Fig. 1.

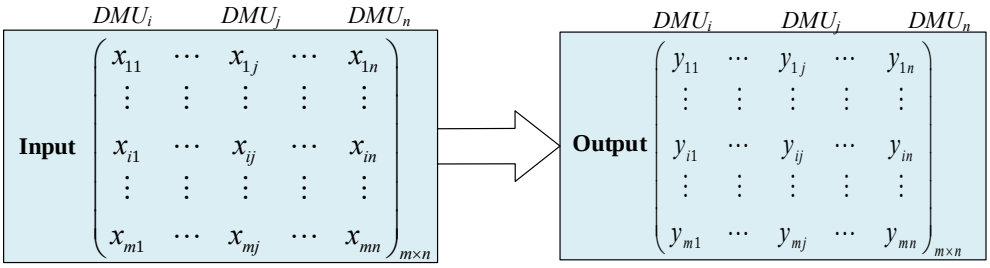

Fig. 1 Input and output structure of DEA model

The CCR model for evaluating the technical input-efficiency of j_0th DMU (DMU_0) under constant returns to scale (CRS) is represented as Eq. 1.

$$\begin{cases} \max h_0 = \sum_{r=1}^{s} \mu_r y_{r0} \\ s.t. \quad \sum_{i=1}^{m} \omega_i x_{i0} = 1 \\ \qquad \sum_{i=1}^{m} \omega_i x_{ij} - \sum_{r=1}^{s} \mu_r y_{rj} \geq 0;\ j = 1, 2, \cdots n \\ \qquad \omega_i,\ \mu_r \geq 0,\ \forall i, r \end{cases} \tag{1}$$

DEA model for evaluating the input-orientated efficiency of DMU_0 under CRS with non-Archimedean infinitesimal is formulated as Eq. 2. Where $\hat{e}$ = (1, 1, … , 1)$^T \in E^m$ and e = (1, 1, … , 1)$^T \in E^s$ are unit vectors, and ε (ε>0) is a non-Archimedean infinitesimal. S^+ and S^- are slacks, reflecting non-radial improvement between one DMU and its optimal value.

$$\begin{cases} \min\ [\theta - \varepsilon(\hat{e}^T S^- + e^T S^+)] \\ s.t. \quad \sum_{j=1}^{n} X_j \lambda_j + S^- = \theta X_0 \\ \qquad \sum_{j=1}^{n} Y_j \lambda_j - S^+ = Y_0 \\ \qquad \lambda_j \geq 0,\ j = 1, 2, \cdots, n \quad S^- \geq 0,\ S^+ \geq 0 \end{cases} \tag{2}$$

Slack based measure can give us more management information about improvement. We construct the model under CRS, offering us an overall efficiency, which includes technical efficiency under variable returns to scale (VRS) and scale effect.

The structure of conventional DEA can be viewed as a “black box” as described in Fig. 2. The “black box” has an input node and an output node, which represents the input and output of data structure respectively. In the box, it requires no assumption on the appearance of the frontier surface as well as it makes no hypothesis concerning the internal operations of a decision making unit.

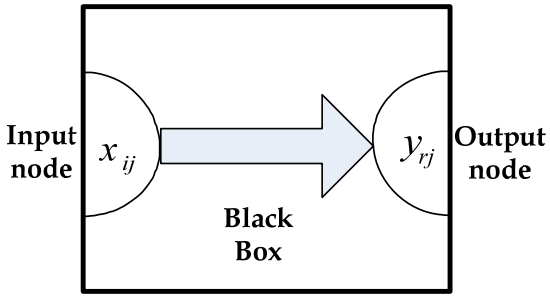

Fig. 2 The structure of conventional DEA

3. Network DEA with Intermediate Products Structure

As an extension of the "black box" model above, Färe and Grosskopf [8] suggested a kind of network DEA model with chain links and non-assignable inputs illustrated in Fig. 3. This framework involves several sub-DMUs which linked one by one as a chain structure. Tone et al. [10, 11] developed the chain network structure by applying it in the dynamic network DEA using a slack-based measure.

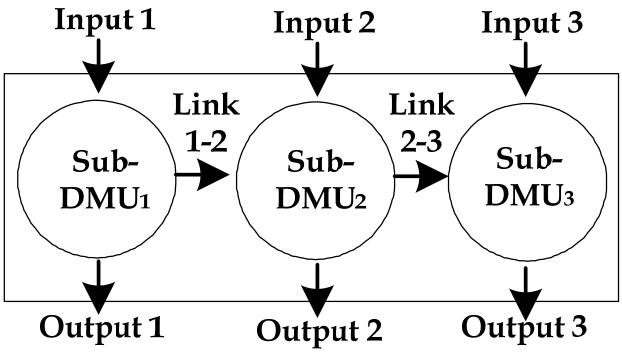

Fig. 3 The structure of network DEA with chain links and non-assignable inputs

In this work, we put forward a network structure with intermediate products between any two sub-DMUs. What's more, the inputs can be assigned to sub-DMUs as well. We suppose that there are n homogeneous DMUs involving s sub-DMUs P_l (l=1, 2, …, s). The input and output of DMU_j is X_j^0 and Y_j^e respectively. Input of the sub-DMUs received from input node are X_j^{0l} and X_j^{0r}, which are assignable and, output of the sub-DMUs exported to output node are Y_j^{le} and Y_j^{re}. There are input and output between two sub-DMUs which we call intermediate products. $\tilde{X}_j^{rl}$ is defined as input from P_r to P_l, $\tilde{Y}_j^{lr}$ is set up as output from P_l to P_r, where, $r, l \in \{1, 2, \cdots, s\}, r \neq l$. We can deduce that $\tilde{X}_j^{rl} = \tilde{Y}_j^{rl}$ and $\tilde{X}_j^{lr} = \tilde{Y}_j^{lr}$. Fig.4 can help us understand the structure of network DEA with intermediate products and assignable inputs.

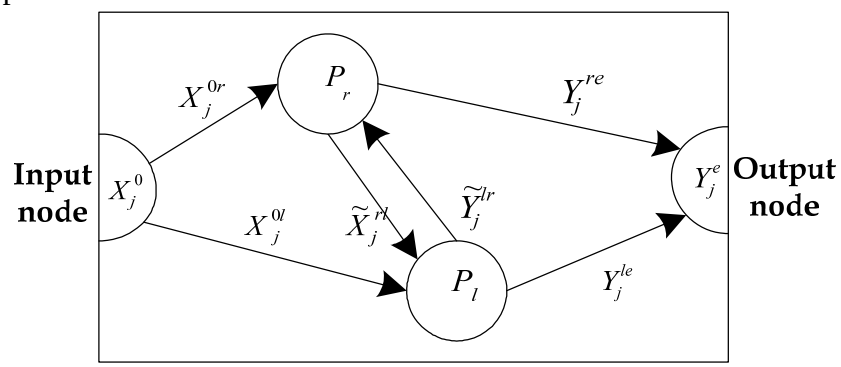

Fig. 4 The structure of network DEA with intermediate products and assignable inputs

When we take intermediate products between sub-DMUs into consideration in DEA, we can formulate network DEA as shown in Eq. 3. The dual model of Eq. 3 can be transformed as Eq. 4.

Where, θ is the efficiency of DMU_0, which reflects the overall efficiency of network DEA with intermediate products. θ is proportional to the performance of the evaluated DMU.

$$(P_{Network})\begin{cases}\max\,(\mu^{eT}Y_0^{\,e})=V_P\\ \omega^{0lT}X_j^{\,0l}+\sum_{r=1,r\neq l}^{s}\tilde{\omega}^{rlT}\tilde{X}_j^{\,rl}-\mu^{leT}Y_j^{\,le}-\sum_{r=1,r\neq l}^{s}\tilde{\mu}^{lrT}\tilde{Y}_j^{\,lr}\geq 0,\\ j=1,2,\cdots n;\ l=1,2,\cdots,s;\ \omega^{0T}X_0^{\,0}=1,\ \omega^{0}\geq\omega^{0l}\\ \tilde{\mu}^{rl}\geq\tilde{\omega}^{rl},\omega^{0l}\geq 0,\ \tilde{\omega}^{rl}\geq 0,\mu^{e}\geq 0,\ \mu^{le}\geq 0,\tilde{\mu}^{rl}\geq 0\end{cases}\tag{3}$$

$$(D_{Network})\begin{cases}\min\,\theta=V_D\\ \sum_{l=1}^{s}X_0^{\,0l}=\theta X_0^{\,0},\ \sum_{j=1}^{n}\lambda_j^{\,l}X_j^{\,0l}+S^{0l-}=X_0^{\,0l},\ \sum_{j=1}^{n}\lambda_j^{\,l}\tilde{X}_j^{\,rl}+\tilde{S}^{rl-}=\tilde{X}_0^{\,rl}\\ \sum_{j=1}^{n}\lambda_j^{\,l}Y_j^{\,le}-S^{le+}=Y_0^{\,le},\ \sum_{j=1}^{n}\lambda_j^{\,l}\tilde{Y}_j^{\,lr}-\tilde{S}^{lr+}=\tilde{Y}_0^{\,lr},\ l,r\in\{1,2,\cdots,s\},l\neq r\\ \lambda_j\geq 0,j=1,2,\cdots,n;\ X_0^{\,0l}\geq 0,S^{0l-}\geq 0,\tilde{S}^{rl-}\geq 0,\ S^{le+}\geq 0,\tilde{S}^{lr+}\geq 0\end{cases}\tag{4}$$

4. Numerical Example

In this section, a numerical example is given to illustrate the operational procedure of network DEA with intermediate structure and assignable input. To this end, ten DMUs with one input node, one output node and three sub-DMUs are considered. We can describe the network DEA structure of this example as shown in Fig. 5.

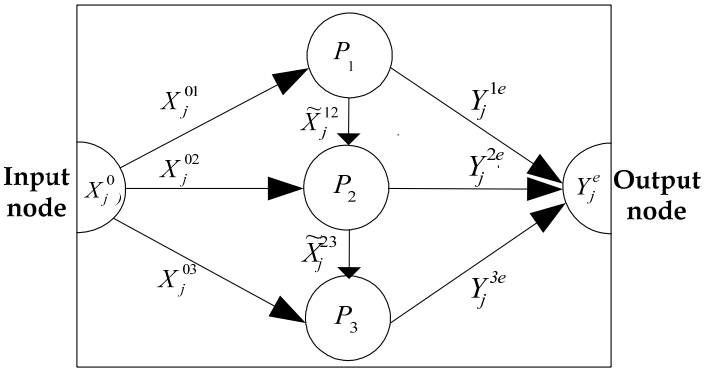

Fig. 5 The structure of network DEA with three sub-DMUs

It is worth mentioning that there are intermediate products between sub-DMU_1 and sub-DMU_2, as well as between sub-DMU_2 and sub-DMU_3. The inputs, outputs and intermediate products for these ten DMUs have been listed in Tab. 1.

Table 1. Input and Output Data of Ten DMUs with Three Sub-DMUs

DMU_j	Input	sub-DMU_1			sub-DMU_2			sub-DMU_3	
	X_j^{0}	X_j^{01}	$\tilde{X}_j^{12}$	Y_j^{1e}	X_j^{02}	$\tilde{X}_j^{23}$	Y_j^{2e}	X_j^{03}	Y_j^{3e}
1	4509	2531	4523	178	1202	809	183	776	947
2	6688	4781	7923	174	1099	501	174	808	529
3	8385	6514	8152	166	1072	597	168	799	547
4	5018	2641	5746	171	1388	655	162	989	636
5	6277	4330	5407	160	1148	708	184	799	851
6	4781	3267	4639	178	1027	291	178	487	392
7	7375	5457	6836	168	1092	580	172	826	530
8	5461	3725	6703	177	1042	768	181	694	895
9	6079	4372	5498	168	911	376	177	796	392
10	5019	3206	4052	172	916	667	178	897	667

By running Eq. 3, we can calculate the whole efficiencies of the network-structure DEA of each DMU, which has three sub-DMUs. The intermediate products of each two sub-DMUs were considered in the model as it can open the "black box" by reveal the connection and interaction between sub-DMUs. We may get the overall efficiencies of the ten DMUs with network structure as shown in Tab. 2.

Table 2. Overall Efficiency of Ten DMUs with Three Sub-DMUs

DMU$_j$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Efficiency θ_j	1	0.71	0.74	0.86	0.82	0.90	0.79	0.83	0.80	0.85

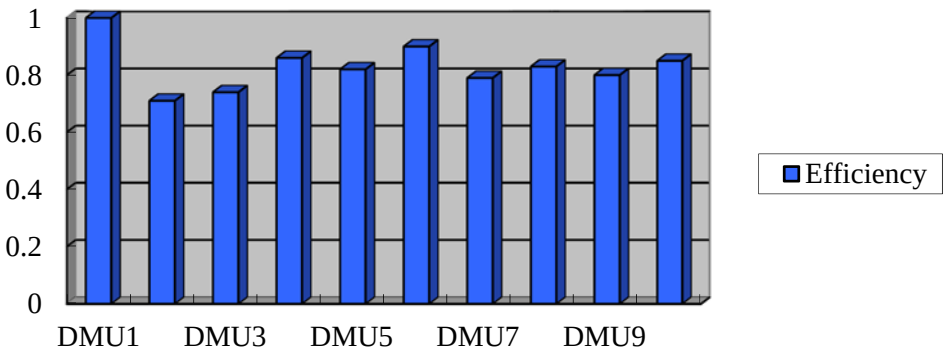

Fig. 6 the efficiencies of the ten DMUs with intermediate products of three sub-DMUs

Fig. 6 reflects the network DEA efficiency for each DMU with three sub-DMUs of intermediate products, from which we can conclude that the network DEA efficiency of the observed DMUs vibrate irregularly. The overall network efficiencies of network DEA can be finally provided in Tab. 2, which can be compared in Fig. 6. We can know that DMU_1 performs best, and DMU_2 displays worst. The ranking sequence shows that $DMU_1 > DMU_6 > DMU_4 > DMU_{10} > DMU_8 > DMU_5 > DMU_9 > DMU_7 > DMU_3 > DMU_2$.

5. Conclusion

In this paper, we put forward a kind of network DEA model, taking intermediate products structure between sub-DMUs into consideration. Thus, the kind of system measures the relative efficiencies of a set of DMUs with network structure of intermediate products between sub-DMUs with assignable inputs. The efficiency reflects overall network efficiency during the one same discrete time spans, showing more practical meaning. Moreover, improved information between inefficient DMUs and optimum values can be provided to decision makers by means of non-Archimedean infinitesimal and slacks of DMUs. Notably, the "black box" DEA model set up above adapts to the structure of network DEA by formulating interaction between sub-DMUs. However, other DEA models can give more importance to more in recent years. For example, super-efficiency DEA can help to solve unsatisfactory differentiation of whole dynamic network efficiencies. Further work based on the structure proposed in this paper could extend the comprehensive dynamic network model to include dynamic pattern as well as other different network structures.

Acknowledgments

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References

[1]. Charnes, W.W. Cooper, and E. Rhodes, “Measuring the efficiency of decision making units,” European Journal of Operational Research, 2 (1978), pp. 429-444.
[2]. Lozano Sebastian, Gutierrez Ester, “A slacks-based network DEA efficiency analysis of European airlines”, Transportation Planning and Technology, 37 (2014), pp. 623-637.
[3]. Chang Tsung-Sheng, Tone Kaoru, Wei Quanling, “Ownership-specified network DEA models”, Annals of Operations Research, 214 (2014), pp. 73-98.
[4]. Maghbouli Mahnaz, Amirteimoori Alireza, Kordrostami Sohrab, “Two-stage network structures with undesirable outputs: A DEA based approach”, Measurement, 48 (2014), pp. 109-118.
[5]. Tavana Madjid, Mirzagoltabar Hadi, Mirhedayatian Seyed Mostafa, Saen Reza Farzipoor, Azadi Majid, “A new network epsilon-based DEA model for supply chain performance evaluation”, Compters & Industrial Engineering, 66 (2013), pp. 501-513.
[6]. R. Färe, R. Grabowski, S. Grosskopf, S. Kraft, “Efficiency of a fixed but allocable input: A nonparametric approach”, Economics Letters, 56 (1997), pp. 187-193.
[7]. Mickael Löthgren, Magnus Tambour, “Productivity and customer satisfaction in Swedish pharmacies: A DEA network model”, European Journal of Operational Research, 115 (1999), pp. 449-458.
[8]. R. Färe, R. Grabowski, “Network DEA”, Socio-Economic Planning Sciences, 34 (2000), pp. 35-49.
[9]. Tone Kaoru, Tsutsui Miki, “Network DEA: A slacks-based measure approach”, European Journal of Operational Research, 197 (2009), pp. 243-252.
[10]. Kaoru Tone, Miki Tsutsui, “Dynamic DEA: A slacks-based measure approach”, Omega, 38 (2010), pp. 145-156.
[11]. Kaoru Tone, Miki Tsutsui, “Dynamic DEA with network structure: A slacks-based measure approach”, Omega, 42 (2014), pp. 124-131.