Comparison theorems between the preconditioned Gauss-Seidel method and the AOR method for M-matrices

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Abstract. In this paper, we study a preconditioned Gauss-Seidel iterative method with the preconditioner which proposed in [1] for solving a linear system whose coefficient matrix is a M-matrix. Some corresponding comparison results between the preconditioned Gauss-Seidel iterative method and the basic AOR iterative method are obtained. Finally, a numerical example is given to illustrate our results.

Introduction

Consider the following linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ nonsingular matrix, x and b are n-dimensional vectors. If A is split into

$$A = M - N,$$

where M is nonsingular, then the basic iterative method for solving the linear system (1) can be expressed in the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \ k = 0,1,\ldots$$

Without loss of generality, let $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and upper triangular matrices of A, respectively. Then the iteration matrix of the classical AOR iterative method (cf. [2]) for solving the linear system (1) is

$$T_{r,w} = (I - rL)^{-1}[(1-w)I + (w-r)L + wU], \tag{2}$$

where w and r are real number with $w \neq 0$.

The iteration matrix of the classical SOR iterative method for solving the linear system (1) is(Note that for $w = r$, we get the SOR iterative method.)

$$T_w = (I - wL)^{-1}[(1-w)I + wU] \tag{3}$$

Then the iteration matrix of the classical Gauss-Seidel iterative method for solving the linear system (1) is(Note that for $w = r = 1$, we get the Gauss-Seidel iterative method.)

$$T_{GS} = (I - L)^{-1}U \tag{4}$$

The spectral radius of the iterative matrix is decisive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. The effective method to decrease the spectral radius is to precondition the linear system, thus we now transform the original system (1) into the preconditioned form

$$PAx = Pb, \tag{5}$$

where $P \in R^{n \times n}$ is a nonsingular matrix. The corresponding basic iterative method is given in general by

$$x^{(k+1)} = M_P^{-1}N_Px^{(k)} + M_P^{-1}Pb, \ k = 0,1,\ldots,$$

where $PA = M_P - N_P$ is a splitting of PA and M_P is nonsingular.

In [1], the authors presented a preconditioned AOR method for consistent systems by using the preconditioner $P_m^a = I + S_m^a$, where

$$S_m{}^a = \begin{bmatrix} 0 & & -a_1 a_{1,k_1} & & \\ 0 & 0 & -a_2 a_{2,k_2} & & \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & & \\ 0 & 0 & \mathbf{L} & \mathbf{O} & -a_{n-1} a_{n-1,k_{n-1}} \\ 0 & 0 & 0 & \mathbf{L} & 0 \end{bmatrix},$$

and a_i is a parameter , $i = 1,2,\mathbf{K},n-1$, $k_i = \min j \in \{ j \big| \max_{j(>i)} |a_{ij}|, \ i < n\}$.

In this paper, we prove theoretically that if A is a nonsingular M-matrix, the spectral radius of precond-itioned Gauss-Seidel iterative method by using the preconditioner is faster than that of AOR method and the SOR iterative method and the Gauss-Seidel iterative method.

Preliminaries

For convenience, we shall now briefly explain some of the terminologies used in the paper. For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n \times n}$ we write $A \ge B$ if $a_{ij} \ge b_{ij}$ holds for $i, j = 1,2,\mathbf{K},n$ .Calling A nonnegative if $A \ge 0$. $(a_{ij} \ge 0; i, j = 1,2,\mathbf{K},n)$. For a square matrix A , $r(A)$ denotes the spectral radius of A .

Definition2.1 A matrix $A = (a_{ij})$ is called a Z-matrix if $a_{ij} \le 0$ for $i \ne j$, Z-matrix is a nonsingular M-matrix, if A is nonsingular and $A^{-1} \ge 0$.

Definition2.2 Let A be a real matrix. The representation $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

(a) regular if $M^{-1} \ge 0$, $N \ge 0$;

(b) weak regular if $M^{-1} \ge 0$, $M^{-1}N \ge 0$.

Lemma 2.3 [4] Let A be a Z-matrix. Then the following statements are equivalent:

(a) A is a nonsingular M-matrix.

(b) All principal sub-matrices of A are non-singular M-matrices.

(c) All principal minors are positive.

Lemma 2.4 [5] Let $A^{-1} \ge 0$ and $A = \overset{\%}{M} - \overset{\%}{N} = M - N$ be two weak regular splittings of A , If $\overset{\%}{M}{}^{-1} \le M^{-1}$ and $\overset{\%}{N} \ge 0$, then $r(\overset{\%}{M}{}^{-1}\overset{\%}{N}) \ge r(M^{-1}N)$.

Lemma 2.5 [6] Let A be a nonsingular M-matrix. Then $r(T_{w,r}) < 1$.

Result and proof

Throughout this paper, we always assume that $a_{ik_i} \ne 0, i = 1,2,\mathbf{K},n-1$. Let $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and upper triangular parts of A , respectively. We denote

$$A_m{}^a = (I + S_m{}^a)A = (I + S_m{}^a)(I - L - U) = I - L - U + S_m{}^a - S_m{}^a L - S_m{}^a U . \tag{6}$$

Then the iterative matrix of the preconditioned Gauss-Seidel iterative method is

$$T_m{}^a = (I - L - D_a' - L_a' F_a)^{-1}(U - S_m{}^a + S_m{}^a U + F_a') , \tag{7}$$

where D_a' , E_a' and F_a' are the diagonal, strictly lower and upper triangular parts of $S_m{}^a L$, respectively.

Theorem 3.1. Let $A = (a_{ij}) \in R^{n \times n}$ be a nonsingular M-matrix. Assume that $0 \le r \le w \le 1, w \ne 0$. Let $T_{r,w}$ and $T_m{}^a$ be defined by (2) and (7), for $a_i \in [0,1]$, $i = 1,2,\mathbf{K},n-1$, $r(T_m{}^a) \le r(T_{w,r}) < 1$.

Proof. Let

$$E_a = I - L - D_a' - L_a' , \ F_a = U - S_m{}^a + S_m{}^a U + F_a' ,$$

As $U \geq S_m{}^a$, it is clear that $F_a \geq 0$. We know that the diagonal elements of E_a are $1 - a_i a_{i,k_i} a_{k_i,i}$, $i \neq n$ and $a_{nn} = 1$. Since A is a nonsingular M-matrix, it is clear that all principal minors of A are positive by Lemma 2.3 .Since all the diagonal entries of A , we have $1 - a_{i,k_i} a_{k_i,i} > 0$. As $a_i \in [0,1]$, $i = 1,2,\mathbf{K},n-1$, we have $1 - a_i a_{i,k_i} a_{k_i,i} > 0$. This implies that all the diagonal entries of E_a are positive, so we know that E_a is a nonsingular M-matrix by Lemma 2.3. Thus we have $E_a{}^{-1} \geq 0$ by Definition 2.1 and then $E_a{}^{-1} F_a \geq 0$. Notice that $I + S_m{}^a$ is nonsingular and let

$$M_a = (I + S_m{}^a)^{-1} E_a, \; N_a = (I + S_m{}^a)^{-1} F_a,$$

then

$$M_a{}^{-1} = E_a{}^{-1}(I + S_m{}^a) \geq 0, \; M_a{}^{-1} N_a = E_a{}^{-1} F_a \geq 0.$$

Since

$$A_m{}^a = (I + S_m{}^a)A = E_a - F_a,$$

we obtain

$$A = M_a - N_a,$$

and $A = M_a - N_a$ is a weak regular splitting of A by Definition 2.2.

As

$$A = \frac{1}{w}(I - rL) - \frac{1}{w}[(1-w)I + (w-r)L + wU],$$

let

$$M_{w,r} = \frac{1}{w}(I - rL), \; N_{w,r} = \frac{1}{w}[(1-w)I + (w-r)L + wU],$$

since $0 \leq r \leq w \leq 1$, $w \neq 0$, we have

$$M_{w,r} \geq 0, \; N_{w,r} \geq 0.$$

So $A = M_{w,r} - N_{w,r} = M_a - N_a$ be two weak splitting of A and $N_{w,r} \geq 0$.

By $0 \leq r \leq w \leq 1$, $w \neq 0$, we obtain

$$\frac{1}{w}(I - rL) = \frac{1}{w}I - \frac{r}{w}L \geq I - L \geq I - L - D'_a - L'_a = E_a,$$

it is clear that

$$M_{w,r} \geq E_a,$$

and therefore

$$M_{w,r}{}^{-1} \leq E_a{}^{-1} \leq E_a{}^{-1}(I + S_m{}^a) = M_a{}^{-1}.$$

So by Lemma 2.4, we have

$$r(T_m{}^a) \leq r(T_{w,r}).$$

Thus from Lemma 2.5, we know $r(T_{w,r}) < 1$, so we obtain

$$r(T_m{}^a) \leq r(T_{w,r}) < 1.$$

Corollary3.2. Let $A = (a_{ij}) \in R^{n \times n}$ be a nonsingular M-matrix. Assume that $0 \leq r \leq w \leq 1, w \neq 0$.
Let T_w and $T_m{}^a$ be defined by (3) and (7), for $a_i \in [0,1]$, $i = 1,2,\mathbf{K},n-1$, $r(T_m{}^a) \leq r(T_w) < 1$.

Proof. We can have the conclusion immediately by let $w = r$ in Theorem 3.1.

Corollary3.3. Let $A = (a_{ij}) \in R^{n \times n}$ be a nonsingular M-matrix. Assume that $0 \leq r \leq w \leq 1, w \neq 0$.
Let T_{GS} and $T_m{}^a$ be defined by (4) and (7), for $a_i \in [0,1]$, $i = 1,2,\mathbf{K},n-1$, $r(T_m{}^a) \leq r(T_{GS}) < 1$.

Proof. We can have the conclusion immediately by let $w = r = 1$ in Theorem 3.1.

Numerical example

In this section, we give a numerical example to illustrate the comparison theorem of the preconditioned Gauss-Seidel iterative method and the AOR iterative method for solving the linear system (1), where A is an nonsingular M- matrix.

If the coefficient matrix A of (1) is given by

$$A = \begin{bmatrix} 1.000 & -0.097 & -0.103 & -0.271 & -0.268 \\ -0.223 & 1.000 & -0.324 & -0.165 & -0.289 \\ -0.002 & -0.354 & 1.000 & -0.390 & -0.642 \\ -0.007 & -0.291 & -0.393 & 1.000 & -0.106 \\ -0.257 & -0.181 & -0.234 & -0.177 & 1.000 \end{bmatrix},$$

and the parameters w, r and $a_i, i = 1,2,\mathbf{K}, n-1$ according to the conditions imposed in Theorem 3.1. For the AOR iterative method, with $w = 0.8$ and $r = 0.7$, then $r(T_{w,r}) = 0.9704$. If we take $a_1 = 0.213$, $a_2 = 0.655, a_3 = 0.964$ and $a_4 = 0.727$, then we have $r(T_m^a) = 0.9308 < r(T_{w,r}) = 0.9704 < 1$.

From the above numerical results, we obtain the results are in concord with Theorem 3.1.

Acknowledgements

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