N-fold Darboux Transformation for a Generalized Variable-Coefficient Coupled Hirota-Maxwell-Bloch System

Yushan Xue and Jing Chen*
School of mathematics and statistics, Central University of Finance and Economics, Beijing 100081, China
*Corresponding author.

Abstract—A generalized variable-coefficient coupled Hirota-Maxwell-Bloch system is investigated, which can describe the ultrashort optical pulse propagation in a variable-coefficient nonlinear, dispersive fiber doped with two-level resonant atoms. Based on the obtained Lax pair, N-fold Darboux transformation (DT) of the system is constructed. One-, two- and three-soliton solutions are derived by virtue of the obtained DT, and compiled into the determinant forms

Keywords—dispersive fiber; lax pair; N-fold darboux transformation

I. INTRODUCTION

In recent years, the investigations of the nonlinear dynamics of variable-coefficient systems have attracted certain attention because those systems are considered to be more realistic than their corresponding homogeneous counterparts [1–4]. In this paper, accompany with the self-induced transparency (SIT) effect, a generalized variable-coefficient coupled Hirota-MB (H-MB) system is proposed [5, 6],

\[
\begin{align*}
\omega p_t &= 2b_1(z)q_p - 2ib_2(z)p \\
\eta_{\beta t} &= (z)(qp^* + q^*p) \\
q(z, t) &= \text{the slowly varying envelope axial field}, \\
p(z, t) &= \text{the measure of the polarization of the resonant medium}, \\
\eta(z, t) &= \text{the extent of the population inversion}, \\
\omega &= \text{the frequency, and the asterisk denotes the complex conjugate.}
\end{align*}
\]

II. DT FOR SYSTEM (1)

The Lax pair corresponding to System (1) can be given as [7–9]

\[
\Phi_t = U\Phi, \quad \Phi_z = V\Phi \quad (2.1a,b)
\]

where

\[
U = \begin{pmatrix} \lambda & \beta \overline{q} \\ \overline{q} \alpha & -\lambda \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & V_2 \\ V_2 & -V_1 \end{pmatrix} = \begin{pmatrix} \lambda \overline{V}_1 + \overline{\lambda} \overline{V}_2 + \lambda \overline{V}_2 + \overline{V}_1 \end{pmatrix},
\]

\[
\Phi = (\Phi_1, \Phi_2)^T \text{is the vector eigenfunction, } T \text{ denotes the transpose of the vector, } \lambda \text{ is the spectral parameter, and the matrices } V_0, V_1, V_2 \text{ and } V_3
\]

III. DT FOR SYSTEM (1)

The transformation

\[
\Phi = T\Phi \quad (2.2)
\]

can transform Lax Pair (2.1a, b) into

\[
\Phi_t = U\Phi, \quad U = (T_z + TU)^{-1} \quad (2.3a)
\]

\[
\Phi_z = V\Phi, \quad V = (T_z + TV)^{-1} \quad (2.3b)
\]

From (2.3a) and (2.3b), one can get

\[
\tilde{U}_z - \tilde{V}_z + [\tilde{U}, \tilde{V}] = T(U_z - V_z + [U, V])T^{-1} \quad (2.4)
\]

Suppose

\[
T = T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (2.5)
\]

where

\[
A = \lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k, \quad B = \sum_{k=0}^{N-1} B_k \lambda^k \quad (2.6)
\]

\[
C = \sum_{k=0}^{N-1} C_k \lambda^k, \quad D = \lambda^N + \sum_{k=0}^{N-1} D_k \lambda^k \quad (2.7)
\]

and

\[
V_0 = \begin{pmatrix} -4\alpha_1 & 0 \\ 0 & 4\alpha_1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -2\alpha_1 & -4\alpha_1 \beta_q \\ 4\alpha_1 \beta_q^* & 2\alpha_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -\alpha_1 - 2\alpha_1 \beta_q^* q_q - 2\beta_q^* (\alpha_q + \alpha_q) \\ 2\beta_q^* (\alpha_q + \alpha_q) & -\alpha_1 + 2\alpha_1 \beta_q^* q_q \end{pmatrix}
\]

\[
V_3 = \begin{pmatrix} A_\eta - \frac{1}{2}i\alpha_q q_q + -\alpha_q \beta_q^* (q^*q_q - q_q^*) - B_1, -\beta_q \end{pmatrix}
\]

with

\[
A_1 = -\beta_1 \alpha_1 / (2(\lambda + t_0)\beta_1)
\]

\[
B_1 = (2\alpha_1 \beta_q q_q^* q_q + \alpha_q q_q + i\alpha_q q_q + \alpha_q q_q^*)
\]

The transformation

\[
\Phi = T\Phi \quad (2.2)
\]
\[
\sum_{k=0}^{N-1} (A_k + B_k \alpha_j) = -\alpha_j^N, \quad \sum_{k=0}^{N-1} (C_k + D_k \alpha_j) = -\alpha_j^N
\]  
(2.8)

with
\[
\alpha_j = \frac{\phi_2(\lambda_j) - \gamma_j \psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j \psi_1(\lambda_j)} \quad (1 \leq j \leq 2N)  
\]  
(2.9)

where \( \phi(\lambda) = \begin{bmatrix} \phi_1(\lambda), \phi_2(\lambda) \end{bmatrix}^T \) and \( \psi(\lambda) = \begin{bmatrix} \psi_1(\lambda), \psi_2(\lambda) \end{bmatrix}^T \) are two basic solutions of Lax Pair (2.1a, b), \( \lambda_k \) and \( \gamma_j (\lambda_k \neq \gamma_j \text{ as } k \neq j) \) are parameters suitably chosen such that the determinant of coefficients for Eq. (2.8) is non-zero, thus, \( A_k, B_k, C_k \) and \( D_k \) are uniquely determined by Eq. (2.8).

From Eq. (2.5), one can see that det\( T(\lambda) \) is a \( 2N \) polynomial of \( \lambda \), and
\[
\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j)
\]
From Eq. (2.8),
\[
A(\lambda_j) = -\alpha_jB(\lambda_j)C(\lambda_j) = -\alpha_jB(\lambda_j)  \quad \text{(2.10)}
\]

Therefore
\[
\det T(\lambda_j) = 0
\]

Thus, \( \lambda_j (1 \leq j \leq 2N) \) are \( 2N \) roots for the determinant of matrix \( T(\lambda) \), i.e.,
\[
\det T(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j)  \quad \text{(2.11)}
\]

Proposition 1. The matrix \( \tilde{U} \) defined by Eq. (2.3a) has the same form as \( U \)
\[
\tilde{U} = \begin{bmatrix} \lambda & \beta_1 \tilde{q}^* \\ -\beta_1 \tilde{q} & -\lambda \end{bmatrix}  \quad \text{(3.1)}
\]

where the transformation from the old potential \( q \) to new one \( \tilde{q} \) is given by
\[
\tilde{q} = q - \frac{2}{\beta_1} B_{N-1}  \quad \text{(3.2)}
\]

Proof. Let \( T^{-1} = T^* / \det T \) and
\[
(T_r + TU)T^* = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{bmatrix}  \quad \text{(2.14)}
\]

\( f_{11}(\lambda) \) and \( f_{22}(\lambda) \) are \( (2N + 1) \)-th order polynomials in \( \lambda \), and \( f_{12}(\lambda) \) and \( f_{21}(\lambda) \) are \( 2N \) th-order polynomials in \( \lambda \). It can be obtained from Eqs. (2.1a, b) and (2.9).

\[
\alpha_j = -\beta_1 \tilde{q}^* - 2\lambda \alpha_j - \beta_1 \beta_2 \alpha_j^2  
\]

It can be verified that all \( \lambda_j (1 \leq j \leq 2N) \) are roots of \( f_{12}(\lambda) (k, j = 1, 2) \). Noting Eq. (2.11), we can prove that
\[
\det(T | f_{kj}(\lambda)) \quad (k, j = 1, 2)
\]

Therefore, together with Eq. (2.14), we have
\[
(T_r + TU)T^* = (\det T)P(\lambda)  \quad \text{(2.15)}
\]

where
\[
P(\lambda) = \begin{bmatrix} P_{11}(\lambda) + P_{12}(\lambda) \\ P_{21}(\lambda) + P_{22}(\lambda) \end{bmatrix}  \quad \text{(3.1)}
\]

Eq. (2.16) can be written as
\[
(T_r + TU)T^* = (\det T)P(\lambda)  \quad \text{(2.17)}
\]

Comparing the coefficients of \( \lambda^{N+1} \) and \( \lambda^N \) in Eq. (2.17), we can obtain
\[
P_{11}(\lambda) = -P_{22}(\lambda) = 1, \quad P_{12}(\lambda) = \beta_1 q - 2B_{N-1} = \beta_1 \tilde{q}  \quad \text{(3.2)}
\]

From Eqs. (2.15) and (2.18), one can see that \( P(\lambda) = \tilde{U} \), the proof is completed.

Similarly, we can prove that the matrix \( \tilde{V} \) has the same form as \( V \) under Transformations (2.2) and (2.13). In summary, we arrive at the following theorem.

**Theorem 1.** The solutions \( (q, p, \eta) \) of System (1) are mapped into their new solutions \( (\tilde{q}, \tilde{p}, \tilde{\eta}) \) under \( DT(2.2) \).

**III. SOLUTIONS AND APPLICATIONS IN INHOMOGENEOUS ERBIUM-DOPED OPTICAL BERS**

In this section, we discuss the DT for System (1) and give its explicit solutions. First, we take
\[
\lambda_{2j} = -\lambda_{2j-1}^*, \quad \gamma_{2j} = -\gamma_{2j-1}^*, \quad 1 \leq j \leq N  
\]

and
\[
\alpha_{2j} = -\alpha_{2j-1}, \quad D_k^* = -A_k, \quad C_k^* = B_k, \quad 1 \leq k \leq N - 1  
\]

In this way, the solutions for Eqs. (2.8) and (2.9) reduce to
\[
\sum_{k=0}^{N-1} (A_k + \alpha_{2j-1} B_k) \lambda_{2j-1}^k = -\lambda_{2j-1}^N  \quad \text{(3.4)}
\]

\[
\sum_{k=0}^{N-1} (-\alpha_{2j-1} A_k + B_k) \lambda_{2j-1}^k = -\alpha_{2j-1}^* \lambda_{2j-1}^N, \quad 1 \leq j \leq N  
\]

with
\[ \alpha_{j-1} = \frac{\phi_2(\lambda_{2j-1}) - \gamma_{2j-1}\psi_2(\lambda_{2j-1})}{\phi_2(\lambda_{2j-1}) - \gamma_{2j-1}\psi_1(\lambda_{2j-1})} \] (3.6)

IV. CONCLUSIONS

In this paper, we have investigated a generalized inhomogeneous coupled H-MB system, i.e., System (1), which describes the ultrashort optical pulse propagation in an inhomogeneous nonlinear, dispersive fiber doped with two-level resonant atoms. Based on the obtained Lax Pair (2.1a, b), we have constructed the N-fold DT for System (1) and compiled the obtained solutions into the determinant forms. Our analysis and results will be of certain value in realizing soliton-type pulse propagation in an inhomogeneous erbium-doped fiber with the higher-order effects.

ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation of China (Grant No.11447233), Social Science Foundation of Beijing (Grant No.15JGC184), Education Teaching Reform Fund 2016 of CUFE.

REFERENCES