

# The (G'/G)-Expansion Method for the Sine-Gordon Equation, Sinh-Gordon Equation and Liouville Equation

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**Abstract.** In this paper, the (G'/G)-expansion method is applied to find the exact solutions for Sine-Gordon equation, Sinh-Gordon equation and Liouville equation. The (G'/G)-expansion method is an effective method in investigating exact traveling wave solutions to nonlinear evolution equations (NLEEs) in the field of applied mathematics, mathematical physics and engineering.

## Introduction

The nonlinear evolution equation  $u_{xt} - h(u) = 0$  (1.1) has important applications in many scientific fields such as nonlinear optics, solid state physics and quantum field theory, where  $u = u(x, t)$  is the unknown function of the space variable  $x$  and time  $t$ , and the function  $h(u)$  may be

$$h(u) = \sin(mu), \sinh(mu), e^{mu}, \dots \quad (1.2)$$

which correspond respectively to the sine-Gordon equation[1]

$$u_{xt} = \sin(mu), \quad (1.3)$$

Sinh-Gordon equation[2]

$$u_{xt} = \sinh(mu), \quad (1.4)$$

Liouville equation[3]

$$u_{xt} = e^{mu}. \quad (1.5)$$

The Sine-Gordon equation is Lax integrable and has nontrivial prolongation structures, which plays an important role in many scientific fields and nonlinear optics. The sinh-Gordon equation is also a Lax integrable system and possesses similarity reductions to third Painlevé equation. It is worth noting that by using the identity  $e^{mu} = \sinh(mu) + \cosh(mu)$ , Liouville equation becomes a combined sinh-cosh-Gordon equation[4]

$$u_{xt} = \sinh(mu) + \cosh(mu). \quad (1.6)$$

These equations have been investigated by different methods, and some exact solutions are derived [5-8]. However, there are still many other exact solutions to be found. In this paper, we will derive more new exact solutions based on (G'/G) -expansion method.

## Description of the (G'/G)-expansion Method

First, we simply describe the (G'/G)-expansion method. For the equation

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (2.1)$$

where  $P$  is a polynomial in its arguments. The transformation  $u(x, t) = f(\xi)$ ,  $\xi = kx - ct$ , reduces Eq.(2.1) to the ordinary differential equation

$$P(f, kf', cf', k^2 f'', kcf'', c^2 f'', \dots) = 0, \quad (2.2)$$

where  $f = f(\xi)$  and the prime denotes derivative with respect to  $\xi$ . We assume that the solution of Eq.(2.2) can be expressed by a polynomial in  $(G'/G)$  as follows:

$$f = \sum_{i=1}^n a_i \left( \frac{G'}{G} \right)^i + a_0, \quad (2.3)$$

where  $G = G(\xi)$  is the solution of the auxiliary linear second-order ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \quad (2.4)$$

where  $G' = \frac{dG}{d\xi}$ ,  $G'' = \frac{d^2G}{d\xi^2}$ ,  $a_n \neq 0, \dots, a_1, a_0, \lambda$  and  $\mu$  are constants to be determined later.

Using the general solutions of Eq.(2.4), we have

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi)}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + C_2 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}{C_1 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + C_2 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \quad V \quad (2.5)$$

where  $C_1, C_2$  are arbitrary constants.

### Exact Traveling Wave Solution of the Equations

Now, we will find the traveling wave solutions of the equations for Eq.(1.3), we employ the transformation:

$$y(x, t) = e^{imu}, \quad (3.1a)$$

for Eq.(1.4) and Eq.(1.5) we employ the transformation:

$$y(x, t) = e^{mu}, \quad (3.1b)$$

where  $i^2 = -1$  from Eq.(3.1), we get

$$u = \frac{1}{m} \operatorname{arccosh} \frac{y^2 + 1}{2y}, \quad (3.2a)$$

and

$$u = \frac{1}{m} \operatorname{arccosh} \frac{y^2 + y^{-2}}{2}. \quad (3.2b)$$

Using the conversion formula, we transform (1.3) and (1.4) into the following equation

$$2yy_{xt} - 2y_x y_t - my^3 + my = 0, \quad (3.3)$$

and Eq.(1.5) is equivalent to the following equation

$$yy_{xt} - y_x y_t - my^3 = 0. \quad (3.4)$$

Now, we will find the traveling wave solutions of the Eq.(3.3) and Eq.(3.4) instead of Eq.(1.3), Eq.(1.4) and Eq.(1.5) by using the  $(G'/G)$ -expansion method.

### Application of the Method to the Eq. (3.3)

In order to obtain the exact traveling wave solutions of the Eq.(3.3), let

$$y(x, t) = f(\xi), \quad \xi = kx - ct, \quad (3.5)$$

then Eq.(3.3) become:

$$-2ckff'' + 2ck(f')^2 - mf^3 + mf = 0. \quad (3.6)$$

in order to balance the term  $f^3$  and  $ff''$  in Eq.(3.6), we get  $n=2$ . Therefore, the solution of Eq(3.6) takes the form:

$$f = a_2 \left( \frac{G'}{G} \right)^2 + a_1 \left( \frac{G'}{G} \right) + a_0, \quad a_2 \neq 0. \quad (3.7)$$

Substituting (3.7) into (3.6) and setting the coefficients of  $\left( \frac{G'}{G} \right)^i$ ,  $0 \leq i \leq 6$ , to zero, we derive a set of algebraic equations for  $a_0$ ,  $a_1$ ,  $a_2$ ,  $c$ ,  $k$  and  $m$ :

$$2ck(a_1^2\mu^2 - a_0a_1\lambda\mu - 2a_0a_2\mu^2) - ma_0^3 + ma_0 = 0, \quad (3.8a)$$

$$2ck(a_1^2\lambda\mu + 2a_1a_2\mu^2 - a_0a_1\lambda^2 - 2a_0a_1\mu - 6a_0a_2\lambda\mu) - 3ma_0^2a_1 + ma_1 = 0, \quad (3.8b)$$

$$2ck(a_1a_2\lambda\mu + 2a_2^2\mu^2 - 4a_0a_2\lambda^2 - 3a_0a_1\lambda - 8a_0a_2\mu) - 3ma_0^2a_2 - 3ma_0a_1^2 + ma_2, \quad (3.8c)$$

$$2ck(-2a_1a_2\mu - a_1^2\lambda - a_1a_2\lambda^2 + 2a_2^2\mu\lambda - 2a_0a_1 - 10a_0a_2\lambda) - 6ma_0a_1a_2 - ma_1^3 = 0, \quad (3.8d)$$

$$2ck(-5a_1a_2\lambda - a_1^2 - 6a_0a_2) - 3ma_0a_2^2 - 3ma_1^2a_2 = 0, \quad (3.8e)$$

$$2ck(-4a_1a_2 - 2a_2^2\lambda) - 3ma_1a_2^2 = 0, \quad (3.8f)$$

$$-4cka_2^2 - ma_2^3 = 0. \quad (3.8g)$$

Solving the set of algebraic equations by use of Maple, we get the following results:

$$\text{case1: } a_0 = \frac{-\lambda^2}{\lambda^2 - 4\mu}, \quad a_1 = \frac{-4\lambda}{\lambda^2 - 4\mu}, \quad a_2 = \frac{-4}{\lambda^2 - 4\mu}, \quad c = \frac{m}{k\lambda^2 - 4k\mu}. \quad (3.9)$$

$$\text{case2: } a_0 = \frac{\lambda^2}{\lambda^2 - 4\mu}, \quad a_1 = \frac{4\lambda}{\lambda^2 - 4\mu}, \quad a_2 = \frac{4}{\lambda^2 - 4\mu}, \quad c = -\frac{m}{k\lambda^2 - 4k\mu}. \quad (3.10)$$

### Case 1

When  $\lambda^2 - 4\mu > 0$ , we obtain hyperbolic function solution:

$$f_{11} = - \left( \frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})} \right)^2. \quad (3.11)$$

When  $\lambda^2 - 4\mu < 0$ , the trigonometric function solution of Eq. (3.3) is:

$$f_{12} = \left( \frac{-C_1 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})}{C_1 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})} \right)^2, \quad (3.12)$$

where  $\xi = kx - ct$ ,  $c = \frac{m}{k\lambda^2 - 4k\mu}$ ,  $C_1$ ,  $C_2$  and  $k$  are arbitrary constants.

In view of Eqs.(3.2), we get the exact explicit traveling wave solution of the sine-Gordon equation are as follows:

$$u_{111} = \frac{1}{m} \operatorname{arccosh} \frac{(f_{11})^2 + 1}{-2f_{11}}, \quad \text{and} \quad u_{112} = \frac{1}{m} \operatorname{arccosh} \frac{(f_{12})^2 + 1}{2f_{12}}.$$

and the exact traveling wave solution of the sine-Gordon equation are as follows:

$$u_{211} = \frac{1}{m} \operatorname{arccosh} \frac{(f_{11})^2 + (f_{11})^{-2}}{2} \quad \text{and} \quad u_{211} = \frac{1}{m} \operatorname{arccosh} \frac{(f_{12})^2 + (f_{12})^{-2}}{2}.$$

### Case 2

When  $\lambda^2 - 4\mu > 0$ , we obtain hyperbolic function solution:

$$f_{21} = \left( \frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})} \right)^2. \quad (3.13)$$

when  $\lambda^2 - 4\mu < 0$ , the trigonometric function solution of Eq.(3.3) is:

$$f_{22} = - \left( \frac{-C_1 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})}{C_1 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})} \right)^2, \quad (3.14)$$

where  $\xi = kx - ct$ ,  $c = -\frac{m}{k\lambda^2 - 4k\mu}$ ,  $C_1, C_2$  and  $k$  are arbitrary constants.

In view of Eqs.(3.2), we get the exact explicit traveling wave solution of the sine-Gordon equation are as follows:

$$u_{121} = \frac{1}{m} \operatorname{arccos} h \frac{(f_{21})^2 + 1}{2f_{21}} \quad \text{and} \quad u_{122} = \frac{1}{m} \operatorname{arccos} h \frac{(f_{22})^2 + 1}{-2f_{22}}.$$

### Application of the Method to the Eq. (3.4)

In order to obtain the exact traveling wave solutions of the Eq (3.4), let

$$y(x, t) = f(\xi), \quad \xi = kx - ct, \quad (3.15)$$

then Eq (3.4) become:

$$-ckff'' + ck(f')^2 - mf^3 = 0. \quad (3.16)$$

Similarly, we get  $n=2$ , suppose Eq.(3.16) has the following formal solutions:

$$f = a_2 \left( \frac{G'}{G} \right)^2 + a_1 \left( \frac{G'}{G} \right) + a_0, \quad a_2 \neq 0. \quad (3.17)$$

Then we derive a set of algebraic equations for  $a_0, a_1, a_2, c, k$  and  $m$ :

$$ck(a_1^2 \mu^2 - a_0 a_1 \lambda \mu - 2a_0 a_2 \mu^2) - ma_0^3 = 0, \quad (3.18a)$$

$$ck(a_1^2 \lambda \mu + 2a_1 a_2 \mu^2 - a_0 a_1 \lambda^2 - 2a_0 a_1 \mu - 6a_0 a_2 \lambda \mu) - 3ma_0^2 a_1 = 0, \quad (3.18b)$$

$$ck(a_1 a_2 \lambda \mu + 2a_2^2 \mu^2 - 4a_0 a_2 \lambda^2 - 3a_0 a_2 \lambda - 8a_0 a_2 \mu) - 3ma_0^2 a_2 - 3ma_0 a_1^2 = 0, \quad (3.18c)$$

$$ck(-2a_1 a_2 \mu - a_1^2 \lambda - a_1 a_2 \lambda^2 + 2a_2^2 \mu \lambda - 2a_0 a_1 - 10a_0 a_2 \lambda) - 6ma_0 a_1 a_2 = 0, \quad (3.18d)$$

$$ck(-5a_1 a_2 \lambda - a_1^2 - 6a_0 a_2) - 3ma_0 a_2^2 - 3ma_1^2 a_2 = 0, \quad (3.18e)$$

$$ck(-4a_1 a_2 - 2a_2^2 \lambda) - 3ma_1 a_2^2 = 0, \quad (3.18f)$$

$$-2cka_2^2 - ma_2^3 = 0. \quad (3.18g)$$

Solving the set of algebraic equations, we get

$$a_0 = \frac{-2\mu}{m}, \quad a_1 = \frac{-2\lambda}{m}, \quad a_2 = \frac{-2}{m}, \quad c = \frac{1}{k}. \quad (3.19)$$

When  $\lambda^2 - 4\mu > 0$ , we obtain hyperbolic function solution:

$$f_{31} = -\frac{\lambda^2 - 4\mu}{2m} \left( \frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi}) + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu\xi})} \right)^2 + \frac{\lambda^2 - 4\mu}{2m}. \quad (3.20)$$

When  $\lambda^2 - 4\mu < 0$ , the trigonometric function solutions of Eq (3.4) can be obtained as follow:

$$f_{32} = \frac{\lambda^2 - 4\mu}{2m} \left( \frac{-C_1 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})}{C_1 \cos(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi}) + C_2 \sin(\frac{1}{2}\sqrt{4\mu - \lambda^2\xi})} \right)^2 + \frac{\lambda^2 - 4\mu}{2m}, \quad (3.21)$$

where  $\xi = kx - ct$ ,  $c = \frac{m}{k\lambda^2 - 4k\mu}$ ,  $C_1, C_2$  and  $k$  are arbitrary constants.

Then we get the exact traveling wave solution of the Liouville equation:

$$u_{331} = \frac{1}{m} \operatorname{arccos} h \frac{(f_{31})^2 + 1}{2f_{31}} \quad \text{and} \quad u_{331} = \frac{1}{m} \operatorname{arccos} h \frac{(f_{32})^2 + 1}{-2f_{32}}.$$

This study shows that the  $(G'/G)$ -expansion method is quite efficient and practically well suited in finding exact solutions for many equations. We hope that they will be useful for further studies in applied sciences.

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