Oscillation Criteria for a Class of Certain Half-linear Emden-Fowler Functional Differential Equations of Neutral Type

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Abstract: Some oscillation criteria for the half-linear Emden-Fowler functional differential equations of neutral type of the form

$$\left(r(t) |z'(t)|^{\alpha-1} z'(t) \right)' + q(t) |x(\sigma(t))|^{\beta-1} x(\sigma(t)) = e(t), t \geq t_0$$

are established, where $z(t) = x(t) + p(t)x(\tau(t))$, $\alpha > 0, \beta > 0$ are arbitrary constants, some earlier results are the special cases of ours since the function $e(t)$ is introduced to the equation.

Introduction

This paper consider half-linear Emden-Fowler functional differential equations of neutral type

$$\left(r(t) |z'(t)|^{\alpha-1} z'(t) \right)' + q(t) |x(\sigma(t))|^{\beta-1} x(\sigma(t)) = e(t), t \geq t_0, \quad (1.1)$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $\alpha > 0, \beta > 0$.

The following conditions are always valid, if not stated.

(A₁) $r(t) \in C^1([t_0, \infty), R)$, $r(t) > 0$, $r'(t) \geq 0$.

(A₂) $p(t), q(t) \in C([t_0, \infty), R)$, $0 \leq p(t) \leq 1$, $q(t) \geq 0$.

(A₃) $\tau(t) \in C([t_0, \infty), R)$, $\tau(t) \leq t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

(A₄) $\sigma(t) \in C^1([t_0, \infty), R)$, $\sigma(t) > 0$, $\sigma'(t) > 0$, $\sigma(t) \leq t$, $\lim_{t \rightarrow \infty} \sigma(t) = \infty$.

(A₅) $e(t) \in C^1([t_0, \infty), R)$, $e(t) \leq 0$.

Definition

The function $x(t) \in C^1([T_x, \infty), R)$, $T_x \geq t_0$, is called a solution of the equation (1.1), if it has the properties of $r(t) |z'(t)|^{\alpha-1} z'(t) \in C^1([T_x, \infty), R)$, and it satisfies the equation in $[T_x, \infty)$, moreover if it has arbitrarily large zeros for $T \geq T_x$, then we call $x(t)$ is a oscillatory solution of equation (1.1), otherwise it is called nonoscillatory. If each solution of (1.1) is oscillatory, we call equation (1.1) is oscillatory. This paper only consider the non-trivial solution of equation (1.1).

Main Results

Theorem 2.1 Assume that $\int_{t_0}^{\infty} \left(\frac{1}{r(t)} \right)^{\frac{1}{\alpha}} dt = \infty$, if there exists a function

$\rho(t) \in C^1([T_0, \infty), (0, \infty))$, such that for any $M > 0$ and $m \in (0, 1)$,

$$\int_{t_0}^{\infty} \left[\rho(t) \bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^{\lambda+1} \frac{r(\lambda(t))}{(\lambda+1)^{\lambda+1}} \left(\frac{\rho(t)}{m\sigma'(t)} \right)^{\lambda} \right] dt = \infty, \quad (2.1)$$

where $\bar{p}(t) = q(t)(1 - p(\sigma(t)))^{\beta}$, $\lambda = \min\{\alpha, \beta\}$,

$$\lambda(t) = \begin{cases} \sigma(t), & \beta \geq \alpha \\ t, & \alpha > \beta \end{cases}, \quad m = \begin{cases} 1, & \alpha = \beta, \\ 0 < m \leq 1, & \alpha \neq \beta. \end{cases}$$

Then Eq. (1.1) is oscillatory.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$, without loss of generality, we assume that $x(t) > 0$.

From ([6](2.7)) we have

$$\left(r(t)(z'(t))^{\alpha} \right)' + \bar{p}(t) z^{\beta}(\sigma(t)) \leq 0, \quad (2.2)$$

which follows that

$$\left(r(t)(z'(t))^{\alpha} \right)' + \bar{p}(t) z^{\beta}(\sigma(t)) \leq e(t), \quad (2.3)$$

where $\bar{p}(t) = q(t)[1 - p(\sigma(t))]^{\beta}$,

then we have

$$\bar{p}(t) z^{\beta}(\sigma(t)) \leq e(t) - \left[r(t)(z'(t))^{\alpha} \right]'. \quad (2.4)$$

Define

$$w(t) = \rho(t) \frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))}. \quad (2.5)$$

Obviously, $w(t) > 0$, and $r(t)(z'(t))^{\alpha}$ is nonincreasing,

$$\begin{aligned} w'(t) &= \rho'(t) \frac{w(t)}{\rho(t)} + \rho(t) \frac{\left[r(t)(z'(t))^{\alpha} \right]'}{z^{\beta}(\sigma(t))} - \rho(t) \frac{\left[r(t)(z'(t))^{\alpha} \right] \beta z^{\beta-1}(\sigma(t)) z'(\sigma(t)) \sigma'(t)}{z^{2\beta}(\sigma(t))} \\ &\leq -\rho(t) \bar{p}(t) + \rho'(t) \frac{w(t)}{\rho(t)} + \frac{e(t) w(t)}{r(t)(z'(t))^{\alpha}} - \rho(t) \frac{\left[r(t)(z'(t))^{\alpha} \right] \beta z^{\beta-1}(\sigma(t)) z'(\sigma(t)) \sigma'(t)}{z^{2\beta}(\sigma(t))}. \end{aligned}$$

which follows that

$$w'(t) \leq -\rho(t) \bar{p}(t) + \left[\frac{\rho'(t)}{\rho(t)} + \frac{e(t)}{r(t)(z'(t))^{\alpha}} \right] w(t) - \frac{\beta \sigma'(t)}{(\rho(t) r(\sigma(t)))^{\frac{1}{\alpha}}} \left[z(\sigma(t)) \right]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t), \quad (2.6)$$

since $e(t) \leq 0$, there exists $M > 0$ such that,

$$r(t)(z'(t))^{\alpha} \leq \frac{1}{M},$$

$$\frac{e(t)}{r(t)(z'(t))^\alpha} \leq Me(t),$$

which follows that

$$\begin{aligned} w'(t) &\leq -\rho(t)\bar{p}(t) + \left[\frac{\rho'(t)}{\rho(t)} + Me(t) \right] w(t) - \frac{\beta\sigma'(t)}{(\rho(t)r(t))^{\frac{1}{\alpha}}} [z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t) \\ &\leq -\rho(t)\bar{p}(t) + \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right| w(t) - \frac{\beta\sigma'(t)}{(\rho(t)r(t))^{\frac{1}{\alpha}}} [z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t). \end{aligned} \quad (2.7)$$

$z'(t)$ is a monotone increasing function, there exists a constant $m \in (0,1)$, such that

$$[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}} \geq m.$$

Then we have

$$w'(t) \leq -\rho(t)\bar{p}(t) + \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right| w(t) - \frac{m\beta\sigma'(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(t). \quad (2.8)$$

Considering

$$Ax - Bx^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} A^{\lambda+1} B^{-\lambda}, \quad \text{where } B > 0, A \geq 0, x \geq 0, \quad (2.9)$$

then

$$\begin{aligned} w'(t) &\leq -\rho(t)\bar{p}(t) + \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^{\lambda+1} \left(\frac{\lambda m \sigma'(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}} \right)^{-\lambda} \\ &= -\rho(t)\bar{p}(t) + \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^{\lambda+1} \frac{r(\lambda(t))}{(\lambda+1)^{\lambda+1}} \left(\frac{m\sigma'(t)}{\rho(t)} \right)^{-\lambda} \\ &= - \left\{ \rho(t)\bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^{\lambda+1} \frac{r(\lambda(t))}{(\lambda+1)^{\lambda+1}} \left(\frac{\rho(t)}{m\sigma'(t)} \right)^\lambda \right\}. \end{aligned} \quad (2.10)$$

Integrating the above inequality from T to t , we have

$$w(t) \leq w(T) - \int_T^t \left\{ \rho(s)\bar{p}(s) - \left| \frac{\rho'(s)}{\rho(s)} + Me(s) \right|^{\lambda+1} \frac{r(\lambda(s))}{(\lambda+1)^{\lambda+1}} \left(\frac{\rho(s)}{m\sigma'(s)} \right)^\lambda \right\} ds,$$

letting $t \rightarrow \infty$, we have $w(t) \rightarrow -\infty$, which is a contradiction. This completes the proof of Theorem 2.1.

Corollary 2.1 Assume that $e(t) = 0$ and (2.1) hold, other conditions unchanged,

$$\int_{t_0}^{\infty} \left[\rho(t)\bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} \right|^{\lambda+1} \frac{r(\lambda(t))}{(\lambda+1)^{\lambda+1}} \left(\frac{\rho(t)}{m\sigma'(t)} \right)^\lambda \right] dt = \infty. \quad (2.11)$$

then Eq. (1.1) is oscillatory.

Corollary 2.2 Assume that $e(t) = 0$ and $\alpha = \beta$,

$$\int_{t_0}^{\infty} \left[\rho(t) \bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} \right|^{\alpha+1} \frac{r(\sigma(t))}{(\alpha+1)^{\alpha+1}} \left(\frac{\rho(t)}{\sigma'(t)} \right)^{\alpha} \right] dt = \infty \quad (2.12)$$

(2.1) is replaced by (2.12), then Eq. (1.1) is oscillatory.

Corollary 2.3 Assume that $\alpha = 1$, condition (2.1) in Theorem 2.1 is replaced by one of the following form:

(1) If $\beta > 1$,

$$\int_{t_0}^{\infty} \left[\rho(t) \bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^2 \frac{r(\sigma(t)) \rho(t)}{4m\sigma'(t)} \right] dt = \infty; \quad (2.14)$$

(2) If $0 < \beta < 1$,

$$\int_{t_0}^{\infty} \left[\rho(t) \bar{p}(t) - \left| \frac{\rho'(t)}{\rho(t)} + Me(t) \right|^{\beta+1} \frac{r(t)}{(\beta+1)^{\beta+1}} \left(\frac{\rho(t)}{m\sigma'(t)} \right)^{\beta+1} \right] dt = \infty. \quad (2.15)$$

The conclusion of theorem 2.1 is still true.

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