

Comparing Properties of Rising Factorial Function with Properties of Falling Factorial Function

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Abstract. This paper firstly introduces the definition of the falling factorial polynomial and the definition of the rising factorial polynomial, and then proves their properties by analysis methods, finally concludes similar properties of rising factorial function with falling factorial function and different properties of rising factorial function with falling factorial function.

Introduction

The factorial operation is encountered in many areas of mathematics, notably in combinatorics, algebra, and mathematical analysis. The falling factorial polynomial (sometimes called the descending factorial, falling sequential product, lower factorial) is defined:

$$x^{(n)} = \prod_{j=0}^{n-1} (x - j), x \in R, n \in N. \quad (1)$$

Remark 1. From the definition of the falling factorial polynomial, we see that $x^{(0)} = 1$; $x^{(1)} = x$; $x^{(2)} = x(x-1)$; and $x^{(n)} = 0$ when $x - n \in \{\dots, -2, -1\}$; and we have

$$x^{(n)} = \prod_{j=0}^{n-1} (x - j) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} = n! \binom{x}{n}, n \in N, x - n \in R \setminus \{\dots, -2, -1\} \quad (2)$$

Where Γ denotes the special gamma function.

The rising factorial polynomial (sometimes called the Pochhammer function, Pochhammer polynomial, ascending factorial, rising sequential product, upper factorial) is defined

$$x^{\bar{n}} = \prod_{j=0}^{n-1} (x + j), x \in R, n \in N. \quad (3)$$

Remark 2. From the definition of the rising factorial polynomial, we have $x^{\bar{0}} = 1$; $x^{\bar{1}} = x$ and $x^{\bar{2}} = x(x+1)$, and $x^{\bar{n}} = 0$ when $x \in \{\dots, -2, -1\}$; and we have

$$x^{\bar{n}} = \prod_{j=0}^{n-1} (x + j) = \frac{\Gamma(x+n)}{\Gamma(x)} = n! \binom{x+n-1}{n}, x \in R \setminus \{\dots, -2, -1, 0\}, n \in N. \quad (4)$$

Remark 3. From the definitions of the falling and rising factorial polynomials, we have

$$x^{\bar{n}} = (x+n-1)^{(n)} = (-1)^n (-x)^{(n)}. \quad (5)$$

Preliminary Definitions and Properties

Extending the two above definitions from an integer n to an arbitrary real number y , the power

function is defined by [3, 4, 6].

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)} = \binom{x+n-1}{n}, \text{ for } x \in R, x-y \in R, \setminus \{\dots, -2, -1\}, \quad (6)$$

$$x^{\bar{y}} = \frac{\Gamma(x+y)}{\Gamma(x)}, \text{ for } y \in R, x \in R \setminus \{\dots, -2, -1, 0\} \quad (7)$$

We assume that $x^{(y)} = 0$ when $x-y \in \{\dots, -2, -1\}$; and $0^{\bar{y}} = 0, x^{\bar{y}} = 0$ when $x \in \{\dots, -2, -1\}$.

Remark 4. Using the properties of the Gamma function it is easily seen that $x^{(y)} > 0$ when $x > -1, x-y > -1$, and $x^{\bar{y}} > -1$, when $x > 0, x+y > 0$,

We will list some of the properties of the falling factorial function with their proofs.

Lemma 1. ([3], Theorem 2.1.). Assume that the following factorial functions are well defined.

$$\Delta x^{(y)} = yx^{(y-1)}, \quad (8)$$

$$\Delta^k x^{(y)} = \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{(y-k)} \quad (9)$$

$$(x-y)x^{(y)} = x^{(y+1)}, \quad (10)$$

$$x^{(x)} = \Gamma(x+1), \quad (11)$$

$$x^{(y)} \leq r^{(y)}, x \leq r, y < x+1, \quad (12)$$

$$x^{(yz)} \geq \left(x^{(y)}\right)^z, 0 < z < 1, \quad (13)$$

$$x^{(y+z)} = (x-z)^{(y)} x^{(z)}, \quad (14)$$

where $\Delta y(t) = y(t+1) - y(t)$.

Proof. The proof of (8). From (6), we have

$$\begin{aligned} \Delta x^{(y)} &= \frac{\Gamma(x+2)}{\Gamma(x+2-y)} - \frac{\Gamma(x+1)}{\Gamma(x+1-y)} \\ &= \frac{(x+1)\Gamma(x+1)}{\Gamma(x+1-(y-1))} - \frac{\Gamma(x+1-y)\Gamma(x+1)}{\Gamma(x+1-(y-1))} \\ &= y \frac{\Gamma(x+1)}{\Gamma(x+1-(y-1))} = yx^{(y-1)}. \end{aligned} \quad (15)$$

The proof of (9). From (8), we get

$$\Delta^k x^{(y)} = \Delta^{k-1} (\Delta x^{(y)}) = \Delta^{k-1} yx^{(y-1)} = \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{(y-k)}. \quad (16)$$

The proof of (10). From (6), we have

$$x^{(y+1)} = \frac{\Gamma(x+1)}{\Gamma(x+1-(y-1))} = \frac{(x-y)\Gamma(x+1)}{\Gamma(x+1-y)} = (x-y)x^{(y)}. \quad (17)$$

The proof of (11). From (6), we obtain

$$x^{(x)} = \frac{\Gamma(x+1)}{\Gamma(x+1-x)} = \Gamma(x+1). \quad (18)$$

The proof of (12). By Euler's infinite product.

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}. \quad (19)$$

For $x \leq r, y < x+1$, we have

$$\begin{aligned} x^{(y)} &= \frac{\Gamma(x+1)}{\Gamma(x+1-y)} = \frac{x+1-y}{x+1} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^{x+1}}{1 + \frac{x+1}{n}} \frac{1 + \frac{x+1-y}{n}}{\left(1 + \frac{1}{n}\right)^{x+1-y}} \\ &= \frac{x+1-y}{x+1} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^y (x+1-y+n)}{x+1+n} \\ &= \left(1 - \frac{y}{x+1}\right) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^y \left(1 - \frac{y}{x+1+n}\right) \\ &\leq \left(1 - \frac{y}{r+1}\right) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^y \left(1 - \frac{y}{r+1+n}\right) \\ &= r^{(y)}. \end{aligned} \quad (20)$$

The proof of (13). From the log-convexity property of the gamma function.

$$\Gamma(za + (1-z)b) \leq (\Gamma(a))^z (\Gamma(b))^{1-z}, 0 < z < 1, \quad (21)$$

We obtain

$$\begin{aligned} x^{(yz)} &= \frac{\Gamma(x+1)}{\Gamma(x+1-yz)} = \frac{\Gamma(x+1)}{\Gamma(z(x+1-y) + (1-z)(x+1))} \\ &\geq \frac{\Gamma(x+1)}{(\Gamma(x+1-y))^z (\Gamma(x+1))^{1-z}} = \left(x^{(y)}\right)^z. \end{aligned} \quad (22)$$

The proof of (14). From (6), we have

$$\begin{aligned}
 x^{(y+z)} &= \frac{\Gamma(x+1)}{\Gamma(x-z+1-y)} \frac{\Gamma(x-z+1)}{\Gamma(x+1-z)} \\
 &= \frac{\Gamma(x-z+1)}{\Gamma(x-z+1-y)} \frac{\Gamma(x+1)}{\Gamma(x+1-z)} = (x-z)^{(y)} x^{(z)}.
 \end{aligned}
 \tag{23}$$

These complete the proofs.

Main Results

We will list some of the properties of the rising factorial function with their proofs.

Theorem 1. ([3], Theorem 2.1.). Assume that the following factorial functions are well defined.

$$\nabla x^{\bar{y}} = y x^{\bar{y}-1}; \tag{24}$$

$$\nabla^k x^{\bar{y}} = \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{\bar{y}-k} \tag{25}$$

$$(x+y)x^{\bar{y}} = x^{\bar{y}+1} \tag{26}$$

$$x^{\bar{x}} = \frac{\Gamma(2x)}{\Gamma(x)}; \tag{27}$$

$$x^{\bar{y}} \geq r^{\bar{y}}, x \leq r, y < x, \tag{28}$$

$$x^{\bar{y}z} \leq (x^{\bar{y}})^z, 0 < y < 1, \tag{29}$$

$$x^{\bar{y+z}} = (x+z)^{\bar{y}} x^{\bar{z}}; \tag{30}$$

Where $\nabla x(t) = x(t) - x(t-1)$.

Proof. The proof of (24)

$$\begin{aligned}
 \nabla x^{\bar{y}} &= x^{\bar{y}} - (x-1)^{\bar{y}} = \frac{\Gamma(x+y)}{\Gamma(x)} - \frac{\Gamma(x+y-1)}{\Gamma(x-1)} \\
 &= \frac{(x+y-1)\Gamma(x+y-1)}{\Gamma(x)} - \frac{(x-1)\Gamma(x+y-1)}{\Gamma(x)} \\
 &= y \frac{\Gamma(x+y-1)}{\Gamma(x)} = y x^{\bar{y}-1}.
 \end{aligned}
 \tag{31}$$

This completes the proof.

The proof of (25)

$$\nabla^k x^{\bar{y}} = \nabla^{k-1} \nabla x^{\bar{y}} = y \nabla^{k-1} x^{\bar{y}-1} = y(y-1) \dots (y-k+1) x^{\bar{y}-k} = \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{\bar{y}-k}. \tag{32}$$

This completes the proof.

The proof of (26). From (7), we have

$$x^{\overline{y+1}} = \frac{\Gamma(x+y+1)}{\Gamma(x)} = (x+y) \frac{\Gamma(x+y)}{\Gamma(x)} = (x+y)x^{\overline{y}}. \quad (33)$$

This completes the proof.

The proof of (27). From (7), we have

$$x^{\overline{x}} = \frac{\Gamma(2x)}{\Gamma(x)}. \quad (34)$$

This completes the proof.

The proof of (28). By Euler's infinite product.

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}. \quad (35)$$

For $x \leq r, y < x$, we have

$$\begin{aligned} x^{\overline{y}} &= \frac{\Gamma(x+y)}{\Gamma(x)} = \frac{x}{x+y} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^{x+y}}{1 + \frac{x+y}{n}} \frac{1 + \frac{x}{n}}{\left(1 + \frac{1}{n}\right)^x} \\ &= \frac{x}{x+y} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^y (x+n)}{x+y+n} = \frac{1}{1 + \frac{y}{x}} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^y}{1 - \frac{y}{x+n}} \\ &= \frac{1}{1 - \frac{y}{x}} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^y}{1 - \frac{y}{x+n}} \\ &\geq \frac{1}{1 - \frac{y}{r}} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^y}{1 - \frac{y}{r+n}} \\ &= r^{\overline{y}} \end{aligned} \quad (36)$$

This completes the proof.

The proof of (29). From the log-convexity property of the gamma function.

$$\Gamma(za + (1-z)b) \leq (\Gamma(a))^z (\Gamma(b))^{1-z}, 0 < z < 1, \quad (37)$$

We obtain

$$\begin{aligned}
 x^{\overline{yz}} &= \frac{\Gamma(x+yz)}{\Gamma(x)} = \frac{\Gamma(z(x+y)+(1-z)x)}{\Gamma(x)} \\
 &\leq \frac{(\Gamma(x+y))^z (\Gamma(x))^{1-z}}{\Gamma(x)} = (x^{\overline{y}})^z.
 \end{aligned}
 \tag{38}$$

The proof of (30). From (7), we have

$$\begin{aligned}
 x^{\overline{y+z}} &= \frac{\Gamma(x+y+z)}{\Gamma(x)} \frac{\Gamma(x+z)}{\Gamma(x+z)} \\
 &= \frac{\Gamma(x+z+y)}{\Gamma(x+z)} \frac{\Gamma(x+z)}{\Gamma(x)} = (x+z)^{\overline{y}} x^{\overline{z}}.
 \end{aligned}
 \tag{39}$$

These complete the proofs.

Conclusion

Similar properties of rising factorial function with falling factorial function.

$$\begin{aligned}
 \Delta x^{(y)} &= yx^{(y-1)}, \nabla x^{\overline{y}} = yx^{\overline{y-1}}; \\
 \Delta^k x^{(y)} &= \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{(y-k)}, \nabla^k x^{\overline{y}} = \frac{\Gamma(y+1)}{\Gamma(y+1-k)} x^{\overline{y-k}}.
 \end{aligned}$$

Different properties of rising factorial function with falling factorial function $(x-y)x^{(y)} = x^{(y+1)}, (x+y)x^{\overline{y}} = x^{\overline{y+1}};$

$$\begin{aligned}
 x^{(x)} &= \Gamma(x+1), x^{\overline{x}} = \frac{\Gamma(2x)}{\Gamma(x)}; \\
 x^{(y)} &\leq r^{(y)}, x^{\overline{y}} \geq r^{\overline{y}}, x \leq r, y < x; \\
 x^{(yz)} &\geq (r^{(y)})^z, x^{\overline{yz}} \leq (x^{\overline{y}})^z, 0 < z < 1; \\
 x^{(y+z)} &= (x-z)^{(y)} x^{(z)}, x^{\overline{y+z}} = (x+z)^{\overline{y}} x^{\overline{z}}.
 \end{aligned}$$

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